COUNTING AND NUMBERING

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International Mathematical Olympiad (IMO) is an annual six-problem contest for pre-collegiate students and is the oldest of the International Science Olympiads.

The first IMO was held in Romania in 1959. It was initially founded for eastern European countries but eventually other countries participated as well. It has since been held annually, except in 1980. About 90 countries send teams of up to six students, plus one team leader, one deputy leader, and observers.

The paper consists of six problems, with each problem being worth seven points, the total score thus being 42 points. No calculators are allowed. The examination is held over two consecutive days; the contestants have four-and-a-half hours to solve three problems per day. The problems chosen are from various areas of secondary school mathematics, broadly classifiable as geometry, number theory, algebra, and combinatorics. They require no knowledge of higher mathematics such as calculus and analysis, and solutions are often short and elementary. However, they are usually disguised so as to make the process of finding the solutions difficult.

Problem 1. (a) How many numbers are in the sequence

$$15,\ 16,\ 17,\ \ldots,\ 190,\ 191\ ?$$

(b) How many numbers are in the sequence

$$22,\ 25,\ 28,\ 31,\ \ldots,\ 160,\ 163\ ?$$

Solution. To answer the above question in a more general framework we need the following definition:

Definition. An **arithmetic progression** or **arithmetic sequence** is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. This difference between any successive terms is called the **ratio** of the arithmetic progression. For instance, the sequence

$$15,\ 16,\ 17,\ \ldots,\ 190,\ 191$$

is an arithmetic progression wirt ratio 1.

To find the number of the terms in an arithmetic progression we use the formula

$$\frac{\text{last term} - \text{first term}}{\text{ratio}} + 1$$

In our case the total number of terms is

$$\frac{191 - 15}{1} = 176 + 1 = 177 \quad \text{terms}$$

For the second example, the sequence

$$22,\ 25,\ 28,\ 31,\ \ldots,\ 160,\ 163$$

is an arithmetic progression with ratio 2 so the number of terms would be

$$\frac{163-22}{3}=47+1=48$$

Let

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n$$

be an arithmetic progression with n terms and having the ration r. From the above formula we find

$$\frac{a_n - a_1}{r} + 1 = n$$

Hence

$$a_n = a_1 + r(n-1)$$

Another important formula concerns the sum of terms in an arithmetic progression

$$a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}$$

In particular we have

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

(b) $1 + 3 + 5 + \cdots + (2n-1) = n^2$

Other useful formulas are as follows

(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(d) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Problem 2. For any positive integer n find the sum

$$S_n = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$$

Solution. Remark that

$$\begin{aligned} S_n &= 1(1+1) + 2(2+1) + 3(3+1) + \cdots + n(n+1) \\ &= (1^2+1) + (2^2+2) + (3^2+3) + \cdots + (n^2+n) \\ &= (1^2+2^2+3^3+\cdots+n^2) + (1+2+3+\cdots+n) \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right] \\ &= \frac{n(n+1)}{2}\frac{2n+4}{3} \\ &= \frac{n(n+1)(n+2)}{3} \end{aligned}$$

In the similar way one can compute

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1)$$

Problem 3. For any positive integer n find the sum

$$S_n = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2)$$

Solution. The general term in the above sum is

$$k(k+1)(k+2)$$

where $k = 1, 2, 3, \ldots, n$

Remark that

$$k(k+1)(k+2) = k(k^2 + 3k + 2) = k^3 + 3k^2 + 2k$$

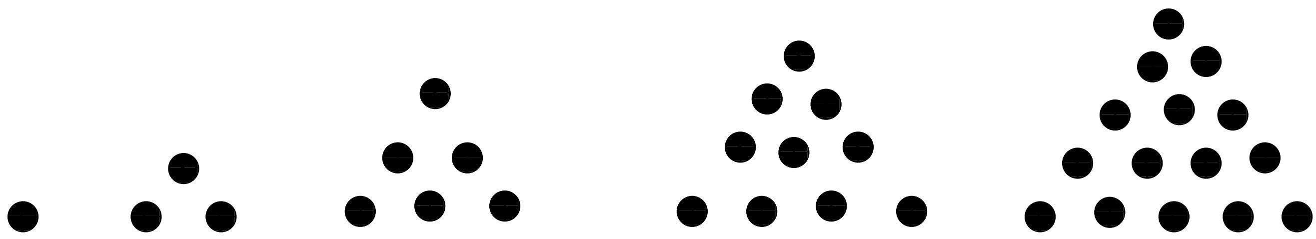
so

$$\begin{aligned}
S_n &= (1^3 + 3 \cdot 1^2 + 2 \cdot 1) + (2^3 + 3 \cdot 2^2 + 2 \cdot 2) + \cdots + (n^3 + 3 \cdot n^2 + 2 \cdot n) \\
&= (1^3 + 2^3 + \cdots + n^3) + 3(1^2 + 2^2 + \cdots + n^2) + 2(1 + 2 + \ldots n) \\
&= \frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + (2n+1) + 2\right] \\
&= \frac{n(n+1)}{2}\frac{n^2 + 5n + 6}{2} \\
&= \frac{n(n+1)(n+2)(n+3)}{4}
\end{aligned}$$

Problem 4. Each of the numbers

$$1 = 1, \quad 3 = 1 + 2, \quad 6 = 1 + 2 + 3, \quad 10 = 1 + 2 + 3 + 4$$

represent the number of balls that can be arranged evenly in an equilateral triangle.



This led the ancient Greeks to call a number **triangular** if it is the sum of consecutive integers beginning with 1.

Prove the following facts about triangular numbers:

(a) If n is a triangular number then $8n + 1$ is a perfect square (Plutarch, circa 100 AD)

(b) The sum of any two triangular numbers is a perfect square (Nicomachus, circa 100 AD)

(b) If n is a triangular number so are the numbers $9n + 1$ and $25n + 3$ (Euler, 1775)

Solution. Remark first that n is a triangular number if there exists a positive integer k such that

$$n = 1 + 2 + 3 + \cdots + k$$

that is,

$$n = \frac{k(k+1)}{2}$$

(a) If $n = \frac{k(k+1)}{2}$ then

$$8n + 1 = 4k(k+1) + 1 = 4k^2 + 4k + 1 = (2k+1)^2$$

(b) Let n and m be two consecutive triangular numbers. Then, there exists $k \geq 1$ such that

$$n = \frac{k(k+1)}{2} \quad \text{and} \quad m = \frac{(k+1)(k+2)}{2}$$

Then

$$n + m = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k(k+1) + (k+1)(k+2)}{2}$$

$$n + m = \frac{(k+1)(2k+2)}{2} = (k+1)^2$$

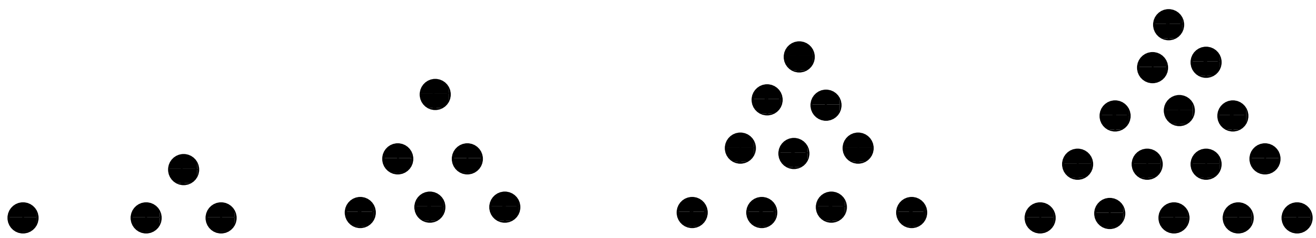
Problem 5. Let t_n be the nth triangular number, that is

$$t_1 = 1, \quad t_2 = 3, \quad t_3 = 6, \quad t_4 = 10, \ldots$$

Prove the formula

$$t_1 + t_2 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}$$

Solution.



We have

$$t_n = \frac{n(n+1)}{2} = \frac{n^2 + n}{2}.$$

Therefore,

$$
\begin{aligned}
t_1 + t_2 + \cdots + t_n &= \frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \cdots + \frac{n^2+n}{2} \\
&= \frac{1^2+2^2+3^2+\cdots+n^2}{2} + \frac{1+2+3+\cdots+n}{2} \\
&= \frac{1}{2}\Big[(1^2+2^2+3^2+\cdots+n^2) + (1+2+\cdots+n)\Big] \\
&= \frac{1}{2}\Big[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\Big] \\
&= \frac{1}{2}\,\frac{n(n+1)}{2}\Big[\frac{2n+1}{3} + 1\Big] \\
&= \frac{1}{2}\,\frac{n(n+1)}{2}\frac{2n+4}{3} \\
&= \frac{n(n+1)(2n+4)}{12} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}
$$

Problem 6. Prove that if an infinite arithmetic progression of positive integers contains a perfect square, then it contains an infinite number of perfect squares.

Solution. Let

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \ldots$$

be an infinite arithmetic progression containing a perfect square, say a^2. Denote by r its ratio. Then, the numbers

$$a^2,\ a^2 + r,\ a^2 + 2r,\ \ldots, a^2 + kr$$

are terms of the above arithmetic progression, $k = 1, 2, 3, \ldots$. In particular the number

$$a^2 + r(2a + r) = a^2 + 2ar + r^2 = (a + r)^2$$

is a perfect square and is another term of the above arithmetic progression. Thus,

$$(a + r)^2,\ (a + r)^2 + r,\ \ldots,\ (a + r)^2 + kr, \ldots$$

are terms of the initial arithmetic progression. As above, it follows that

$$(a + r)^2 + r[2(a + r) + r^2] = (a + 2r)^2$$

is a perfect square and belongs to the initial arithmetic progression. We have obtained so far that $(a + r)^2$, $(a + 2r)^2$ are terms in the progression. Preoceeding similarly we obtain that all the perfect squares

$$(a+r)^2,\ (a+2r)^2,\ \ldots,\ (a+100r)^2, \ldots$$

are terms in the initial arithmetic progression.

Problem 7. Prove that there are no arithmetic progressions of positive integers whose terms are all perfect squares.

Solution. Assume by **contradiction** that there exists positive integers

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \dots$$

such that

$$a_1^2 < a_2^2 < \cdots < a_n^2 < a_{n+1}^2 < \dots$$

is an arithmetic progression. Then, the ratio of it would be

$$r = a_2^2 - a_1^2 = a_3^2 - a_2^2 = \cdots = a_n^2 - a_{n-1}^2 = a_{n+1}^2 - a_n^2 = \dots$$

It follows that

$$(a_n - a_{n-1})(a_n + a_{n-1}) = (a_{n+1} - a_n)(a_{n+1} + a_n), \quad n = 2, 3, 4, \dots$$

Since $a_{n-1} < a_n < a_{n+1}$ we have $a_{n+1} + a_n > a_n + a_{n-1}$ so the above equality yields

$$a_2 - a_1 > a_3 - a_2 > a_4 - a_3 > \cdots > a_n - a_{n-1} > \cdots > 0$$

which is clearly impossible.

Problem 8. Let

$$N = 1234\ldots91011\ldots99100101\ldots20082009$$

be a number obtained by joining all positive integers from 1 to 2009.

(a) How many digist has N?

(b) Remove 9 digits from N such that N becomes as small as possible.

Solution. (a) In writing the number N we use all the numbers from 1 to 2009. We hav thus to count how many numbers of 1,2,3 and 4 digits are joined to write N.

- Numbers of 1 digis: 1,2,...,9 : 9 digits in total
- Numbers of 2 digits: 10,11,12,...,99 in total we have

$$(99 - 9) \times 2 = 180 \quad \text{digits}$$

- Numbers of 3 digits: 100,101,...,999

$$(999 - 99) \times 3 = 2700 \quad \text{digits}$$

- Numbers of 4 digits: 1000,1001,...,2009

$$(2009 - 999) \times 4 = 1010 \times 4 = 4040 \quad \text{digits}$$

In total, N has

$$9 + 180 + 2700 + 4040 = 6929 \quad \text{digits}$$

(b) $N = 12345678910111213...2009$

We remove the digits 2,3,4,5,6,7,8,9, from 10 we remove 1 and from 12 we remove 2.

We obtain

$$M = 1011113141516...2009$$

Problem 9. (a) Find how many numbers from 1 to 1000 are divisible by 7.

(b) Find how many numbers from 1 to 1000 are divisible either by 7 or by 11.

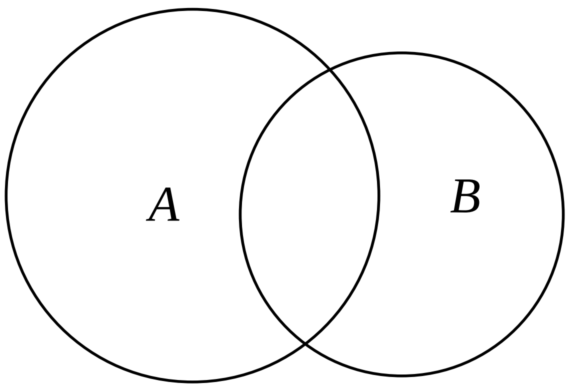
Solution. We start with the following definition

Definition The **integer part** of a number x is the greatest integer that is less or equal to x. It is denoted by $[x]$.

Example $[3.1] = 3$, $[5.76] = 5$ but $[-3.1] = -4$ and $[-5.76] = -6$

(a) In our case, the number of multiples of 7 from $1, 2, 3, \ldots, 100$ equals $\left[\frac{1000}{7}\right] = 142$

(b) Let A be the set of numbers from 1 to 1000 that are divisibly by 7 and let B be the set of numbers from 1 to 1000 that are divisible by 11. Then the set of numbers divisible either by 7 or by 11 is the set $A \cup B$.



We have the formula

$$|A \cup B| = |A| + |B| - |A \cap B|$$

We have seen that $|A| = 142$ and similarly $|B| = \left[\frac{1000}{7}\right] = 90.$ also $A \cap B =$ the set of numbers between 1 to 1000 that are divisible with both 7 and 11, so

$$|A \cap B| = \left[\frac{1000}{77}\right] = 12$$

Therefore $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 99 - 12 = 229$ numbers from 1 to 1,000 are divisible eithr by 7 or by 11.

Problem 10. Which numbers from the sequence 1,2,1,...,1000000 are more: those divisible by 11 but not by 13 or those divisible by 13 but not by 11?

Solution.

Let A be the set of all positive integers between 1 and 1,000,000 that are divisible with 11 and let B be the set of numbers between 1 to 1,000,000 that are divisible by 13.

We are required to find the number of elements of the sets $A \setminus B$ and $B \setminus A$.

Remark that $A \cap B$ represents the set of all positive integers between 1 to 1,000,000 that are divisible by 11 and 13.

By the previous formula we have

$$|A| = \left[\frac{1,000,000}{11}\right] = 90,909 \qquad |B| = \left[\frac{1,000,000}{13}\right] = 76,923$$

$$|A \cap B| = \left[\frac{1,000,000}{143}\right] = 6,993$$

Then

$$|A \setminus B| = |A| - |A \cap B| = 90,909 - 6,993 = 83,916$$

and

$$|B \setminus A| = |B| - |A \cap B| = 76,923 - 6,993 = 69,930$$

There are more numbers divisible by 11 and not by 13.

Problem 11. A board 9×9 is divided into 81 unit squares. Find the number of squares with sides parallel to the sides of the initial board that contain an integer number of unit squares.

Solution. We have to find the number of 1×1, 2×2, ..., 9×9 squares on the board. First, it is easy to see that the number of 1×1 squares is $9^2 = 81$.

To find the number of 2×2 squares on the board we have to count the number of red unit squares in the figure below as they represent the top-left corner of a possible 2×2 square.

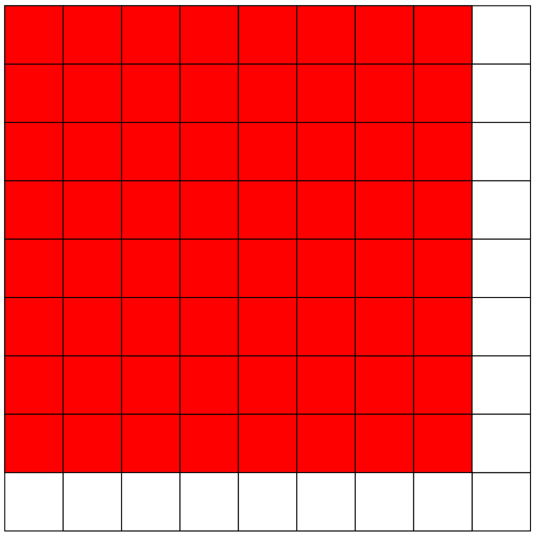


Figure 1. The top left unit square of any 2×2 must be one of the red squares

Therefore, there are 8^2 squares of side lenght 2. Similarly, the number of 3×3 squares is 7^2, the number of 4×4 squares is 6^2,..., the number of 9×9 squares is 1^2. The number of the required squares is

$$1^2 + 2^2 + ... + 8^2 + 9^2 = \frac{9 \cdot 10 \cdot 19}{6} = 285$$

Homework

1. Calculate the sums

(i) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2)$

(ii) $1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 6 + 3 \cdot 5 \cdot 7 + \cdots + n(n+2)(n+4)$

2. Prove that if an infinite arithmetic progression contains a perfect cube, then it contains infinitely many perfect cubes.

3. (a) Find the numbers from the sequence 1,2,3,...,137 that are divisible by 3 or by 5.

(b) (a) Find the numbers from the sequence 1,2,3,...,137 that are divisible either by 3 or by 5 but not by 7.