Trigonometrical identities and inequalities

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1 A review of the trigonometrical functions

These are $\sin, \cos, \& \tan$. These are discussed in the Maynooth Olympiad Manual, which we refer to as MOM! We assume that you know the following addition formulae: for all x, y

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

Letting $x = y$ we deduce the following double-angle formulae: for all x,

$$\sin(2x) = 2 \sin x \cos x, \ \sin 0 = 0;$$

$$\cos(2x) = \cos^2 x - \sin^2 x, \ \ \cos 0 = \cos^2 x + \sin^2 x.$$

From the last of these we see that $\cos 0 = \cos^2 0$, whence $\cos 0$ is 0 or 1. But $\cos 0 = 0$ implies that $0 = \cos^2 x + \sin^2 x$, $\forall x$, i.e., $\cos x = \sin x = 0$ for all real x, which is false. So, $\cos 0 = 1$. Hence

$$\cos(2x) = \cos^2 x - \sin^2 x, \ \ 1 = \cos^2 x + \sin^2 x,$$

and so

$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

formulae which are very useful, and important to remember.

Exercise 1 *Show that*

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

as long as $\cos x \cos y \neq 0$. *What's the result if* $\cos x \cos y = 0$*?*

Exercise 2 *Verify that*

1. $$2\cos(x+y)\cos(x-y) = \cos(2x) + \cos(2y);$$

2. $$2\sin(x+y)\sin(x-y) = \cos(2y) - \cos(2x);$$

3. $$\sin(2x) + \sin(2y) = 2\sin(x+y)\cos(x-y);$$

4. $$\sin(2x) - \sin(2y) = 2\cos(x+y)\sin(x-y).$$

1.1 Some values of these functions

In a right-angled triangle ABC, which has its right-angle at C,

$$\sin A = \frac{BC}{AB} = \frac{a}{c},\ \cos A = \frac{AC}{AB} = \frac{b}{c},$$

and

$$\sin B = \frac{AC}{AB} = \frac{b}{c},\ \cos B = \frac{BC}{AB} = \frac{a}{c}.$$

Applying these formulae to the right-angled triangle ABC, where $a = 1, b = \sqrt{3}, c = 2$, we infer that A has 30 degrees or $\pi/6$ radians, B has 60 degrees or $\pi/3$ radians, and C has 90 degrees or $\pi/2$ radians, whence

$$\sin\frac{\pi}{6} = \frac{1}{2},\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Similarly,

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3} = \frac{1}{2},$$

and

$$\sin\frac{\pi}{2} = \frac{AB}{AB} = 1.$$

Applying these formulae to the right-angled triangle ABC, where $a = 1 = b, c = \sqrt{2}$, we infer that A has 45 degrees or $\pi/4$ radians, B has 45 degrees or $\pi/4$ radians, and C has 90 degrees or $\pi/2$ radians, whence

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Exercise 3 *Evaluate*

$$\tan\frac{\pi}{3},\ \tan\frac{\pi}{6},\ \tan\frac{\pi}{4}.$$

Is $\tan\frac{\pi}{2}$ *defined?*

1.2 The Cosine Rule, and some consequential formulae

The above formulae tell us that in a *right-angled triangle* we can compute the sines, cosines and tangents of the acute angles, once we know the side lengths of the triangle. In an arbitrary triangle, we can compute the cosine of any one of its angles by using the Cosine Rule: if ABC is a triangle, and a, b, c are the lengths of the sides BC, CA, AB, respectively, then

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},\ \cos B = \frac{c^2 + a^2 - b^2}{2ca},\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

(See Mom, p. 28)Are there similar-type expressions for the sines of the angles? Yes, and they can be determined by using the fact that

$$\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x).$$

Exercise 4 *Show that in any triangle ABC*

$$1 - \cos A = \frac{(a + b - c)(c + a - b)}{2bc},\ 1 + \cos A = \frac{(a + b + c)(b + c - a)}{2bc}.$$

Deduce that

$$\cos\frac{A}{2} = \frac{\sqrt{s(s-a)}}{\sqrt{bc}},\ \sin\frac{A}{2} = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{bc}},$$

where s is the semi-perimeter *of ABC, i.e., $2s = a + b + c$.*

Combining the last pair of these, we see that

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc},\ \tan\frac{A}{2} = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}.$$

2 The area, the inradius and circumradius of a triangle ABC

It's customary to denote these objects by Δ, r, R, respectively. If we need to be specific about the area of a particular triangle ABC, we sometimes write (ABC) in place of Δ. Depending on the data, there are different ways of computing Δ. For example,

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C.$$

Hence, appealing to the formula for $\sin A$, given above,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

This is known as Heron's formula; it's symmetric in the side lengths a, b, c.

Note, too, that

$$2\frac{\Delta}{abc} = \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This is the familiar Sine Rule for a triangle.

Exercise 5 *Prove that*

$$16\Delta^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4.$$

2.1 The Circumcircle

The perpendicular bisectors of the sides of any triangle concur at a point which is equidistant from its vertices. (This is a theorem, which we take for granted; it follows from Ceva's theorem which was discussed last year. Consult your notes from last year; and see MOM p.127.) The point of concurrency is the centre of a circle, called the circumcentre of the triangle, that passes through these points. The radius of this circle is called the circumradius of the triangle; it's denoted by R. Referring to a couple of suitable diagrams—obtained by supposing the circumcentre is an internal point or an external point of the triangle—(for one diagram see MOM p.27) you should be able to argue that

$$\sin A = \frac{a/2}{R} = \frac{a}{2R}, \ \frac{1}{2R} = \frac{\sin A}{a}.$$

Hence, we can compute R, once we know the measure of an angle and the length of the opposite side. If the lengths of the three sides are specified, we can use an alternative formula:

$$\frac{1}{2R} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{abc} = \frac{2\Delta}{abc}.$$

Exercise 6 *Show that in any triangle ABC,*

$$\sin A \sin B \sin C = \frac{8\Delta^3}{(abc)^2}.$$

2.2 The incircle

The inradius of a triangle is a concept dual to the circumradius. It is the radius of the circle that touches the three sides *internally.* (This hints at the possibility that there are exradii associated with the triangle as well. Can you see what these might be?) That such a circle exists is a consequence of the fact that, by Ceva's theorem alluded to above, the angle bisectors concur at a point which is equidistant from the three sides. The radius of this circle is dented by r; it is called the inradius of the triangle.

Given ABC, denote by I the point of concurrency of the angle bisectors. Then, referring to the diagram on p. 32 of MOM, it's clear that the area of ABC is composed of the areas of the three triangles AIB, BIC, CIA, which have sides in common, but are otherwise disjoint. Hence

$$\begin{aligned}
\Delta &= (ABC) \\
&= (AIB) + (BIC) + (CIA) \\
&= \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br \\
&= \frac{r}{2}(a+b+c) \\
&= sr,
\end{aligned}$$

whence

$$r = \frac{\Delta}{s}.$$

Thus knowing the side lengths we can compute r.

Exercise 7 *Show that in any triangle ABC,*

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r\Delta}{abc} = \frac{\Delta^2}{sabc},$$

and

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{\Delta s}{abc} = \frac{\Delta^2}{rabc}.$$

3 The graphs of sine and cosine

Both sin and cos are periodic of periodic 2π:

$$\sin(x+2\pi) = \sin x, \;\; \cos(x+2\pi) = \cos x, \;\forall x.$$

This means that the graphs of both follow a wave pattern, and that we can confine our attention to the parts of the graphs that lie above the interval $[0, 2\pi]$. We cam make the following remarks about the graph of $y = \sin x$.

1. it is anti-symmetric (centrally symmetric ?) about the origin, i.e.,

   $$\sin(-x) = -\sin x, \;\forall x.$$

   In other words, sin is an *odd* function—like the functions $x, x^3, x^5, \ldots$, and linear combinations of these such as $2x - x^3 + 5x^7$;

2. it is symmetric about $x = \pi/2$, i.e.,

   $$\sin(\pi - x) = \sin x, \;\forall x;$$

3. it increases on $[0, \pi/2] \cup [3\pi/2, 2\pi]$, and decreases on $[\pi/2, 3\pi/2]$;

4. the slope of the tangent is positive at any point of the curve over $(0, \pi/2)$, zero at $x = \pi/2$, and negative at any point of the curve over $(\pi/2, \pi)$.

5. the slope of the tangent at any point of the curve decreases on $(0, \pi)$, and increases on $(\pi, 2\pi)$.

6. the arc of the curve over $[0, \pi]$ joining any two points of it lies above the chord connecting these points. In other words, sin is *concave* on this interval.

7. by contrast, sin is *convex* on $[\pi, 2\pi]$: on this interval, the arc joining two points lies below the chord joining them.

Exercise 8 *Noting that* $\cos(\pi/2 - x) = \sin x$*, or otherwise, establish the following properties of the graph of* $\cos$*:*

1. $\cos(-x) = \cos x$*, i.e,* $\cos$ *is* even*, like* $1, x^2, x^4, \ldots$*, and linear combinations of these such as* $2 + x^2 - 4x^4 + 5x^{10}$*;*

2. $\cos$ *decreases on* $[0, \pi]$*, and increases on* $[\pi, 2\pi]$*;*

3. $\cos$ *is concave on* $[0, \pi/2] \cup [3\pi/2, 2\pi]$*, and convex on* $[\pi/2, 3\pi/2]$*.*

4 Some trigonometrical inequalities

In case of a triangle ABC, in Sections 2.1 and 2.2 above, we encountered expressions like the products $\sin A \sin B \sin C$ and $\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$, which are functions of the (measures of the angles) A, B, C which are deemed to be positive numbers satisfying the constraint $A + B + C = \pi$. By allowing these to vary subject to these constraints, the values of the products change. However, since $0 \leq \sin x \leq 1$, for all $x \in [0, \pi]$, and $0 \leq \cos x \leq 1$, for all $x \in [0, \pi/2]$, and at most one angle in a triangle is a right-angle, and none has measure 0 or π, we see that, in any triangle ABC,

$$0 < \sin A \sin B \sin C < 1, \quad 0 < \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} < 1.$$

The questions arise: what are the extreme values of these products? Are these extreme values attained in some triangle?

To answer these and similar questions, we establish a lemma which is a statement about the graph of $y = \log \sin x$ on $(0, \pi)$, namely that it is concave on this interval. (See if you can sketch the graph of this function.)

Lemma 1 *If* $x, y \in [0, \pi]$*, then*

$$\sqrt{\sin x \sin y} \leq \sin \frac{x + y}{2}.$$

There is equality iff $x = y$*.*

Proof. Since $\sin x \geq 0$ for all $x \in [0, \pi]$, the given inequality is equivalent to the statement that

$$2 \sin x \sin y \leq 2 \sin^2 \frac{x+y}{2} = 1 - \cos(x+y) = 1 - \cos x \cos y + \sin x \sin y,$$

i.e.,

$$\sin x \sin y + \cos x \cos y \leq 1, \Leftrightarrow \cos(x-y) \leq 1,$$

which is true. Moreover, the inequality is strict unless $\cos(x-y) = 1$, i.e, unless $x - y$ is an even multiple of π. But $-\pi \leq x - y \leq \pi$. Hence $x = y$.

Here's another approach: since $\sqrt{ab} \leq (a+b)/2$, if $a, b \geq 0$, with equality iff $a = b$, we see that

$$\sqrt{\sin x \sin y} \leq \frac{\sin x + \sin y}{2} = \sin \frac{x+y}{2} \cos \frac{x-y}{2} \leq \sin \frac{x+y}{2},$$

with equality as before.

We can now build on this to deduce a more general inequality.

Lemma 2 *If $x, y, z \in [0, \pi]$, then*

$$\sqrt[3]{\sin x \sin y \sin z} \leq \sin \frac{s+y+z}{3}.$$

There is equality here iff $x = y = z$.

Proof. We introduce another variable $w \in [0, \pi]$ and apply the result just proved to the expression

$$\sqrt[4]{\sin x \sin y \sin z \sin w} = \sqrt{\sqrt{\sin x \sin y}\sqrt{\sin z \sin w}}.$$

Doing so, we see that

$$\begin{aligned} \sqrt[4]{\sin x \sin y \sin z \sin w} &\leq \sqrt{\sin \frac{x+y}{2} \sin \frac{z+w}{2}} \\ &\leq \sin \left(\frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} \right) \\ &= \sin \left(\frac{x+y+z+w}{4} \right). \end{aligned}$$

Moreover, there is equality in the first inequality iff $x = y$ and $z = w$, and in the second iff $x + y = z + w$. Thus, there is equality throughout iff $x = y = z = w$.

We exploit this to get the result we want. Given $x, y, z \in [0, \pi]$ let $w = (x + y + z)/3$. Then $w \in [0, \pi]$, and $w = (x + y + z + w)/4$. Hence by what we've just done

$$\sqrt[4]{\sin x \sin y \sin z \sin w} \leq \sin w, \ \sqrt[3]{\sin x \sin y \sin z} \leq \sin w = \sin\left(\frac{x + y + z}{3}\right),$$

and there is equality iff $x = y = z = w$.

Exercise 9 *More generally, prove that if $x_1, x_2, \ldots, x_8 \in [0, \pi]$, then*

$$\sqrt[p]{\sin x_1 \sin x_2 \cdots \sin x_p} \leq \sin\left(\frac{x_1 + x_2 + \cdots + x_p}{p}\right), \ p = 1, 2, \ldots, 8,$$

with equality iff $x_1 = x_2 = \cdots = x_p$.

As a consequence , we have the following theorems.

Theorem 1 *In any triangle ABC,*

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}.$$

Moreover, the inequality is strict unless ABC is equilateral.

Proof. By Lemma 2,

$$\sin A \sin B \sin C \leq \left(\sin \frac{A + B + C}{3}\right)^3 = \sin^3 \frac{\pi}{3} = (\frac{\sqrt{3}}{2})^3 = \frac{3\sqrt{3}}{8},$$

and there is equality iff $A = B = C = \pi/3$.

Corollary 1

$$\Delta \leq \frac{\sqrt{3}\sqrt[3]{(abc)^2}}{4},$$

with equality iff ABC is equilateral.

Proof. For, by Exercise 5,

$$\sin A \sin B \sin C = \frac{8\Delta^3}{(abc)^2},$$

whence

$$\frac{8\Delta^3}{(abc)^2} \leq \frac{3\sqrt{3}}{8},\ \Delta^3 \leq \frac{3\sqrt{3}(abc)^2}{64} = \left(\frac{\sqrt{3}}{4}\right)^3 (abc)^2,$$

with equality iff ABC is equilateral.

So, knowing the side lengths of a triangle we can obtain a crude upper estimate of its area.

Taking account of the fact that the geometric mean of three positive numbers doesn't exceed their arithmetic mean, we can deduce a relationship between the area of a triangle and its perimeter.

Corollary 2 (Isoperimetric inequality)

$$\Delta \leq \frac{s^2}{3\sqrt{3}},$$

with equality iff ABC is equilateral.

Proof. For, $\sqrt[3]{(abc)^2} = (\sqrt[3]{abc})^2$, and

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3} = \frac{2s}{3},$$

with equality if $a = b = c$.

This tells us that among all triangles with the same perimeter, the equilateral triangle encloses the largest area. This fact extends to any *convex* polygon. In particular, among all rectangles withe same perimeter, the square encloses the largest area.

Exercise 10 *Prove that in any triangle ABC,*

$$\Delta \leq \frac{a^2+b^2+c^2}{4\sqrt{3}},$$

with equality iff ABC is equilateral.

Theorem 2 *Prove that in any triangle ABC,*

$$\sin\frac{A}{2}\sin\frac{A}{2}\sin\frac{A}{2} \leq \frac{1}{8},$$

with equality iff ABC is equilateral.

Proof. Once more by Lemma 2,

$$\sin\frac{A}{2}\sin\frac{A}{2}\sin\frac{A}{2} \le \left(\sin\frac{A+B+C}{6}\right)^3 \sin^3\frac{\pi}{6} = \frac{1}{8},$$

with equality iff $A = B = C = \pi/3$.

Corollary 3 (Euler) *In any triangle ABC, $2r \le R$, with equality iff ABC is equilateral.*

Proof. For

$$\frac{r}{R} = \frac{4\Delta r}{abc} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{2}.$$

Exercise 11 *Prove that*

$$r \le \frac{abc}{8\Delta}.$$

Exercise 12 *Let $x, y \in [0, \pi/2]$. Prove that*

$$\sqrt{\cos x \cos y} \le \cos\frac{x+y}{2},$$

with equality iff $x = y$.

Exercise 13 *Let $x, y, z \in [0, \pi/2]$. Prove that*

$$\sqrt{\cos x \cos y \cos z} \le \cos\frac{x+y+z}{3},$$

with equality iff $x = y = z$.

Exercise 14 *Let ABC be a triangle. Prove that*

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8},$$

with equality iff ABC is equilateral.

Exercise 15 *Prove that, in any triangle ABC,*

$$\Delta \le \frac{3\sqrt{3}abc}{4(a+b+c)},$$

with equality iff ABC is equilateral.