Problem set: The Pigeonhole Principle and the Induction Principle

Irish Mathematical Olympiad Training, 2009/2010

13 February 2010

Problem 1:

Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet in a point. [For example, $f(4) = 11$.] Find a formula for $f(n)$.

Problem 2 (IrMO 1988):

Suppose you are given n blocks, each of which weighs an integral number of pounds, but less than n pounds. Suppose also that the total weight of the n blocks is less than $2n$ pounds. Prove that the blocks can be divided into two groups, one of which weighs exactly n pounds.

Problem 3 (IrMO 1997):

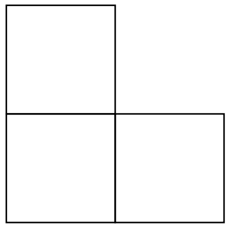
Let A be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with k a nonnegative integer) or there exist two distinct elements $a, b \in A$ such that $a + b$ is a power of 2.

Problem 4:

n people sit in a row of n seats. They are allowed to rearrange themselves such that each person moves by at most one seat. Find the number of ways a_n that they can rearrange.

Problem 5:

Let n be a positive integer. Prove that if one square of a $2^n \times 2^n$ chessboard is removed, the remaining board can be tiled with 3-square tiles of the following shape:



NOTE: The following problems relate to the Fibonacci sequence, which is defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

Problem 6:

Prove that for all $n \geq 1$,

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 .$$

Problem 7:

Prove that for all $n \geq 1$,

$$F_1F_2 + F_2F_3 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2 .$$

Problem 8:

Prove that for all $n \geq 2$,

$$F_nF_{n+1} - F_{n-2}F_{n-1} = F_{2n-1} .$$