TWENTY SECOND IRISH MATHEMATICAL OLYMPIAD

Saturday, 9 May 2009

Second Paper

Time allowed: **Three hours.**

1. Let $p(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer n such that the polynomial $q(x)$ defined by

$$q(x) = p(x+n) - p(x)$$

has integer coefficients.

2. For any positive integer n define

$$E(n) = n(n+1)(2n+1)(3n+1)\cdots(10n+1).$$

Find the greatest common divisor of $E(1)$, $E(2)$, $E(3)$, ..., $E(2009)$.

3. Find all pairs (a, b) of positive integers, such that $(ab)^2 - 4(a+b)$ is the square of an integer.

4. At a strange party, each person knew exactly 22 others.

For any pair of people X and Y who knew one another, there was no other person at the party that they both knew.

For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew.

How many people were at the party?

5. In the triangle ABC we have $|AB| < |AC|$. The bisectors of the angles at B and C meet AC and AB at D and E respectively. BD and CE intersect at the incentre I of $\triangle ABC$.

Prove that $\angle BAC = 60^\circ$ if and only if $|IE| = |ID|$.

Solutions

1. Proposed by Stephen Buckley.

   Solution

   Each term in $p(x)$ is of the form $a_i x^i$, where a_i is rational. Expanding $a_i x^i - a_i(x+k)^i$, we see that k is a factor in all terms. Thus it suffices to pick k to equal the LCM of the denominators of the numbers a_i.

2. Proposed by Marius Ghergu.

   Solution

   Let m be the g.c.d. of $E(1), E(2), E(3), \dots, E(2009)$.

   Since $m|E(1) = 2 \cdot 3 \cdot \ldots \cdot 11$, it follows that any prime divisor of m is less than or equal to 11. Let p be a prime number such that $p|m$. Since $p \leq 11 < 2009$, it follows that $m|E(p) = p(p+1)(2p+1)(3p+1)\cdots(10p+1)$. Remark that $p+1$, $2p+1$, $3p+1$, $\dots$, $10p+1$ are relatively prime to p, so $E(p)$ (and thus m) is divisible by p but not by p^2. We have thus proved that m is not divisible by the square of any prime number.

   Since $m|E(1) = 2 \cdot 3 \cdot \ldots \cdot 11$, it follows that m divides the product of all prime numbers less than or equal to 11, that is, $m|2310$.

   To show that $m = 2310$ it is enough to prove that for all $n \geq 1$, the number $E(n)$ is divisible by 2310.

   Let $n \geq 1$. Then, one of the numbers n or $n+1$ is divisible by 2, so $2|E(n)$. Similarly, one of the numbers n, $n+1$, $2n+1$ is divisible by 3 so $3|E(n)$. Then, one of the numbers $n, n+1$, $2n+1$, $3n+1$, $4n+1$ is divisible by 5 which yields $5|E(n)$. In the same manner we obtain $7|E(n)$ and $11|E(n)$. Therefore $E(n)$ is a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ and so, the g.c.d. is 2310.

3. Proposed by Bernd Kreussler.

   Solution

   If $(ab)^2 - 4(a+b) = x^2$ with positive integers a, b and an integer $x \geq 0$, we have $x < ab$. As $(ab)^2 - (ab-1)^2 = 2ab - 1$ is odd, we even have $x \leq ab - 2$. This implies $(ab)^2 - 4(a+b) \leq (ab-2)^2 = (ab)^2 - 4ab + 4$, from which we obtain

   $$ab \leq a + b + 1. \qquad (1)$$

   After swapping a and b if necessary, we may assume $a \leq b$. If $a \geq 3$, we get $ab \geq 3b \geq a + b + b > a + b + 1$ in contradiction to (1). Hence $a = 2$ or $a = 1$.

   If $a = 1$, we have $b^2 - 4(b+1) = x^2$, which is equivalent to $(b-2-x)(b-2+x) = 8$. Because $(b-2-x)+(b-2+x) = 2b-4$ is even and $b-2-x \leq b-2+x$, the only possibility is $b-2-x=2$ and $b-2+x=4$. This yields $(a,b)=(1,5)$ as the only possible solution with $1=a\le b$.

If $a=2$, we have $4b^2-4(b+2)=x^2$, which is equivalent to $(2b-1-x)(2b-1+x)=9$. Here we have two possibilities. Either $2b-1-x=2b-1+x=3$ or $2b-1-x=1, 2b-1+x=9$. In the first case we obtain $b=2$ and in the second $b=3$. So we have shown that $(a,b)=(2,2)$ and $(a,b)=(2,3)$ are the only possible solutions with $2=a\le b$.

A simple calculation verifies that the five pairs $(1,5),(5,1),(2,2),(2,3)$ and $(3,2)$ indeed satisfy the requirements of the problem.

4. Proposed by Tom Laffey.

   Solution

   Suppose there were n people at the party. For each person P_i at the party, let

   $$S_i=\{j : P_i \text{ knows } P_j\}.$$

   Fix i. We count the number of distinct pairs (j,k) such that $j\in S_i$ and $k\in S_j$. There are $22^2=484$ such pairs in all. There are 22 such pairs with $k=i$. Suppose $k\ne i$. Then P_k is one of the $n-22-1$ people different from P_i that P_i does not know and there are 6 corresponding j for which we must include (j,k) in our count. Hence $484=22+6(n-23)$ and $n=100$.

5. Proposed by Jim Leahy.

   Solution

   Let $\angle BAC=2\alpha, \angle CBA=2\beta$ and $\angle ACB=2\gamma$. Assume first that $2\alpha=\angle BAC=60°$. This implies $2\beta+2\gamma=120°$, i.e. $\beta+\gamma=60°$. Hence, $\angle DIE=\angle BIC=120°$. Therefore, $\angle BAC+\angle DIE=180°$ and the quadrilateral $EIDA$ is cyclic. As AI bisects $\angle BAC$, the chords EI and DI subtend angles of $30°$ at the circumference of the circumcircle of $EIDA$. This implies $|IE|=|ID|$.

Conversely, assume $|IE| = |ID|$. The bisector BD divides CA in the ratio $|AB| : |BC|$. This can easily be seen from the sine rule for the two triangles $\triangle BDA$ and $\triangle BCD$ and using that $\sin(180^\circ - x) = \sin(x)$.

Let $|BC| = a, |CA| = b$ and $|AB| = c$. From $\frac{|CD|}{|DA|} = \frac{a}{c}$ and $|CD| + |DA| = b$ we obtain obtain $|DA| = \frac{bc}{a+c}$. Similarly we get $|AE| = \frac{bc}{a+b}$. Because $|CA| > |AB|$ by assumption, we have $b > c$ and so $\frac{bc}{a+c} > \frac{bc}{a+b}$, hence $|DA| > |AE|$.

Let D' be the reflection of D in AI. Since $|DA| > |AE|$, D' will lie between E and B on AB. Then $\triangle AID \equiv \triangle AID'$, hence $|ID'| = |ID|$ and $\angle ID'A = \angle ADI = 2\gamma + \beta$. Since $|IE| = |ID|$ we have $|IE| = |ID'|$ from which we get $\angle D'EI = \angle ID'A = 2\gamma + \beta$. From $\angle IEA = 2\beta + \gamma$ we obtain now

$$180^\circ = \angle IEA + \angle D'EI = 2\beta + \gamma + 2\gamma + \beta = 3(\beta + \gamma) ,$$

which implies $\beta + \gamma = 60^\circ$. Since $\alpha + \beta + \gamma = 90^\circ$, we get $\alpha = 30^\circ$ and so $\angle BAC = 2\alpha = 60^\circ$.