TWENTY SECOND IRISH MATHEMATICAL OLYMPIAD

Saturday, 9 May 2009

First Paper.

Time allowed: **Three hours.**

1. Hamilton Avenue has eight houses. On one side of the street are the houses numbered 1,3,5,7 and directly opposite are houses 2,4,6,8 respectively. An eccentric postman starts deliveries at house 1 and delivers letters to each of the houses, finally returning to house 1 for a cup of tea. Throughout the entire journey he must observe the following rules. The numbers of the houses delivered to must follow an odd-even-odd-even pattern throughout, each house except house 1 is visited exactly once (house 1 is visited twice) and the postman at no time is allowed to cross the road to the house directly opposite. How many different delivery sequences are possible?

2. Let $ABCD$ be a square. The line segment AB is divided internally at H so that $|AB|.|BH| = |AH|^2$. Let E be the mid point of AD and X be the midpoint of AH. Let Y be a point on EB such that XY is perpendicular to BE. Prove that $|XY| = |XH|$.

3. Find all positive integers n for which $n^8 + n + 1$ is a prime number.

4. Given an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ where each $x_i = +1$ or -1, form a new n-tuple
$$(x_1x_2, x_2x_3, x_3x_4, \ldots, x_nx_1),$$
and continue to repeat this operation. Show that if $n = 2^k$ for some integer $k \geq 1$, then after a certain number of repetitions of the operation, we obtain the n-tuple
$$(1, 1, 1, \ldots, 1)\ .$$

5. Suppose a, b, c are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Prove that
$$a^2b^2c^2 \leq \frac{1}{54},$$
and determine the cases of equality.

Solutions

1. Proposed by **Gordon Lessells**.

   Considering the order in which the houses 1,2,3,4 are visited we have six possibilities

   $$1234, 1243, 1324, 1342, 1423, 1432$$

   For each of these we can find an ordering of 5678 which describes the order in which the houses 5,6,7,8 are visited. In each case there turn out to be two orderings which satisfy the criteria. Intertwining these with the ordering of 1234 gives a total of twelve delivery sequences as follows:

   $$17283546, 18253647, 17254638, 18274536, 16382547, 18352746$$

   $$16354728, 18364527, 16472538, 17452836, 16453827, 17463528$$

   Alternatively,there are six choices for the second and eighth houses to be visited. Each of these gives rise to two possibilities.

2. Proposed by **Jim Leahy**

   Let the square $ABCD$ have length $2a$. Let $|AH| = x$. Then

   $$x^2 = 2a(2a - x)$$

   Hence

   $$x^2 + 2ax = 4a^2$$

   Solving we find $x = \sqrt{5} - 1$.
   Observe that $\triangle BXY$ and $\triangle BEA$ are similar. Hence

   $$\begin{aligned} |XY| &= \frac{|BX|}{\sqrt{5}} \\ &= \frac{3 - \sqrt{5} + \frac{\sqrt{5}-1}{2}}{\sqrt{5}} \\ &= \frac{\sqrt{5} - 1}{2} \end{aligned}$$

   Hence $|XH| = |XY|$.

3. Proposed by **Bernd Kreussler**.

Let $f(x) = x^8 + x + 1$. Numerical values get large very quickly:

$$\begin{aligned} f(1) &= 3 \\ f(2) &= 259 = 7 \times 37 \\ f(3) &= 6565 = 5 \times 13 \times 101 \\ f(4) &= 65541 = 3 \times 7 \times 3121. \end{aligned}$$

These numbers may suggest that $f(n)$ will be a prime number only if $n = 1$. To prove this, we try to factorise the polynomial $x^8 + x + 1$. Progress can be made if it is suspected that $x^2 + x + 1$ is a factor. This can quickly be tested by using a cubic root of unity $\omega \neq 1$. It satisfies $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$, hence $\omega^8 = \omega^2$ from which we directly see $f(\omega) = 0$. Polynomial division gives now easily the factorisation $f(x) = (x^2 + x + 1)(x^6 - x^5 + x^3 - x^2 + 1)$.

Another way to obtain this factorisation is the following. We write $x^8+x+1 = x^8 - x^2 + x^2 + x + 1$ and observe

$$x^8 - x^2 = x^2(x^6 - 1) = x^2(x^3 + 1)(x^3 - 1) = x^2(x^3 + 1)(x - 1)(x^2 + x + 1).$$

This gives $f(x) = x^8 + x + 1 = (x^2 + x + 1)\big(x^2(x^3 + 1)(x - 1) + 1\big)$.
If $n \geq 2$, we have $n^2 + n + 1 \geq 7$ and $n^2(n^3 + 1)(n - 1) + 1 \geq 37$, hence $f(n)$ is not a prime number if $n \geq 2$. As $f(1) = 3$ is a prime number, we conclude that $n = 1$ is the only positive integer for which $n^8 + n + 1$ is a prime number.

4. Proposed by **Donal Hurley**.

First Solution: Use induction on k. Result clear for $k = 1$. Assume it is true for some $k > 1$ and now consider an arbitrary n-tuple $(x_1, x_2, \ldots, x_n)$ of length $n = 2^{k+1}$. Since ${x_i}^2 = 1$ for all i, the second iteration

$$(x_1 \, {x_2}^2 \, x_3, \; x_2 \, {x_3}^2 \, x_4, \; \ldots, \; x_{n-1} \, {x_n}^2 \, x_1, \; x_n \, {x_1}^2 \, x_2)$$

can be written as

$$(x_1 \, x_3, \; x_2 \, x_4, \; x_3 \, x_5, \; \ldots, \; x_{n-1} \, x_1, \; x_n \, x_2)$$

which is the result of the interlacing of the two $(\frac{n}{2})$-tuples

$$(x_1\,x_3,\ x_3\,x_5,\ \ldots,\ x_{n-1}\,x_1) \quad\text{and}\quad (x_2\,x_4,\ x_4\,x_6,\ \ldots,\ x_n\,x_2) \qquad (**).$$

The same rule can be used to obtain the fourth iteration of the original n-tuple by interlacing the second iteration of the two $(\frac{n}{2})$-tuples of $(**)$, the sixth iteration of the original by interweaving the third iterations etc.. Thus $2\,j$ iterations $(j \geq 2)$ of the original n-tuple yields the same result as the interlacing of the j^{th} iterations of the two $(\frac{n}{2})$-tuples of $(**)$. But the induction hypothesis guarantees that these iterations of the $(**)$ tuples consist only of ones for sufficiently large j. Thus we conclude that, for j sufficiently large, $2\,j$ iterations of the original n-tuple gives the n-tuple

$$(1,\ 1,\ \ldots,\ 1)$$

as required.

Second Solution: Throughout, we assume that all subscripts are read "modulo n". For example, if $n = 4$, then x_5 is the same as x_1, x_6 means x_2, etc. More generally, x_{n+i} is to identified with x_i.

Let $(x_{1,r}, x_{2,r}, \ldots, x_{n,r})$ be the rth iterate. So

$$(x_{1,0}, x_{2,0}, \ldots, x_{n,0}) = (x_1, x_2, \ldots, x_n),$$

$$(x_{1,1}, x_{2,1}, \ldots, x_{n,1}) = (x_1x_2, x_2x_3, \ldots, x_nx_1),$$
$$(x_{1,2}, x_{2,2}, \ldots, x_{n,2}) = (x_1x_2^2x_3, x_2x_3^2x_4, \ldots, x_nx_1^2x_2)$$

etc...

Lemma: For all $r \geq 0$,

$$x_{i,r} = \prod_{j=0}^{r} x_{i+j}^{\binom{r}{j}}.$$

Proof: We can prove this by induction on r. The cases $r = 0$ and $r = 1$ are clearly true. Now suppose that $r \geq 2$ and that

$$x_{i,l} = \prod_{j=0}^{l} x_{i+j}^{\binom{l}{j}}$$

for all $l \leq r-1$. Then

$$\begin{aligned}
x_{i,r} &= x_{i,r-1}x_{i+1,r-1} \\
&= \prod_{j=0}^{r-1} x_{i+j}^{\binom{r-1}{j}} \prod_{j=0}^{r-1} x_{i+1+j}^{\binom{r-1}{j}} \\
&= \prod_{j=0}^{r-1} x_{i+j}^{\binom{r-1}{j}} \prod_{j=1}^{r} x_{i+j}^{\binom{r-1}{j-1}} \\
&= x_i \left(\prod_{j=1}^{r-1} x_{i+j}^{\binom{r-1}{j}} x_{i+j}^{\binom{r-1}{j-1}} \right) x_{i+r} \\
&= x_i \left(\prod_{j=1}^{r-1} x_{i+j}^{\binom{r-1}{j}+\binom{r-1}{j-1}} \right) x_{i+r} \\
&= x_i \left(\prod_{j=1}^{r-1} x_{i+j}^{\binom{r}{j}} \right) x_{i+r} \\
&= \prod_{j=0}^{r} x_{i+j}^{\binom{r}{j}}
\end{aligned}$$

as required. QED

Observe that

$$x_{i,n} = x_i \left(\prod_{j=1}^{n-1} x_{i+j}^{\binom{n}{j}} \right) x_{i+n} = \prod_{j=1}^{n-1} x_{i+j}^{\binom{n}{j}}$$

since $x_i x_{i+n} = (x_i)^2 = 1$

Now, we complete the solution by proving...

Lemma. For all $k \geq 1$ and $1 \leq j \leq 2^k - 1$, the binomial coefficient $\binom{2^k}{j}$ is even.

Proof: There are various ways to prove this. For example, we can use the fact that the power of 2 that divides $m!$ is

$$\left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2^2} \right\rfloor + \left\lfloor \frac{m}{2^3} \right\rfloor + \dots$$

Therefore, the power of 2 that divides $\binom{2^k}{j}$ is

$$2^{k-1} + \dots + 1 - \left(\left\lfloor \frac{j}{2} \right\rfloor + \dots + \left\lfloor \frac{j}{2^{k-1}} \right\rfloor \right) - \left(\left\lfloor \frac{2^k - j}{2} \right\rfloor + \dots + \left\lfloor \frac{2^k - j}{2^{k-1}} \right\rfloor \right) \quad (1)$$

However, the expression in (**??**) is strictly bigger than 0 since for all $1 \le j \le 2^k - 1$ and $1 \le s \le k$,

$$2^{k-s} \ge \left\lfloor \frac{j}{2^s} \right\rfloor + \left\lfloor \frac{2^k - j}{2^s} \right\rfloor$$

and the inequality is strict for at least one s between 1 and k (this happens whenever 2^s does not divide j). QED

Thus, if $n = 2^k$ then $x_{i,n} = 1$ for all i.

5. Proposed by **Finbarr Holland**.

First Solution. Since the result is trivial if one of a, b, c is zero, and not all of the nonzero ones can have the same sign, we may suppose without loss of generality that $a, b > 0$ and $c < 0$.

Then $c = -(a + b)$, whence

$$1 = a^2 + b^2 + c^2 = 2(a^2 + ab + b^2),$$

and so

$$a^2 + ab + b^2 = \frac{1}{2}.$$

Hence, by AM–GM,

$$3ab = ab + 2ab \le ab + a^2 + b^2 = \frac{1}{2},$$

i.e., $ab \le 1/6$, with equality if and only if

$$a = b = \frac{1}{\sqrt{6}}.$$

It follows that

$$c^2 = a^2 + 2ab + b^2 = \frac{1}{2} + ab \le \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

with equality as before. So

$$a^2 b^2 c^2 = (ab)^2 (c^2) \le \frac{1}{36} \frac{2}{3} = \frac{1}{54},$$

with equality if and only if

$$a = b = \frac{1}{\sqrt{6}}, c = -\frac{2}{\sqrt{6}}.$$

Finally, removing the sign restriction imposed at the outset, we see that the result holds and that there is equality only when two of a, b, c are equal to $\pm 1/\sqrt{6}$ and the third is equal to $\mp 2/\sqrt{6}$.

Second Solution. First of all,

$$0 = (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca),$$

i.e.,

$$ab + bc + ca = -\frac{1}{2},$$

and so a, b, c are the roots of the cubic

$$x^3 - \frac{1}{2}x - abc = 0.$$

Since this cubic is in normal form $x^3 - px - q = 0$, and the roots of this are real, $27q^2 \leq 4p^3$, with equality if and only if two of the roots are equal, and the third is the negative of one of these, (*)

we may deduce that

$$27(abc)^2 \leq 4(\frac{1}{2})^3 = \frac{1}{2},\ a^2b^2c^2 \leq \frac{1}{54},$$

with equality as before.

Remark. Proof of (*). This follows from the following well-known statement about the discriminant of a cubic in normal form:-

Suppose a, b, c are the roots of the cubic $x^3 - px - q$. Then

$$(a-b)^2(b-c)^2(c-a)^2 = 4p^3 - 27q^2.$$

Alternatively, an examination of the graph of a real cubic reveals that its roots are real if and only if the product of its local extrema is nonnegative. A calculus argument reveals that the extrema occur when $x = \pm\sqrt{p/3}$. Hence the requirement is that

$$(\frac{p\sqrt{p}}{3\sqrt{3}} - \frac{p\sqrt{p}}{\sqrt{3}} - q)(-\frac{p\sqrt{p}}{3\sqrt{3}} + \frac{p\sqrt{p}}{\sqrt{3}} - q) \leq 0 \Leftrightarrow (-\frac{2p\sqrt{p}}{3\sqrt{3}} - q)(\frac{2p\sqrt{p}}{3\sqrt{3}} - q) \leq 0,$$

i.e.,

$$-\frac{4p^3}{27} + q^2 \leq 0,$$

with equality if and only if 0 is either a local max or a local min, in which case the cubic has a double root, etc..