Ireland's Participation in the 60th International Mathematical Olympiad

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The 60th International Mathematical Olympiad (IMO) took place in Bath, United Kingdom, from 11–22 July 2019. A total of 621 students (65 of whom were girls) participated from 112 countries. These numbers make it the IMO with the largest participation so far.

The Irish delegation consisted of six students (see Table 1) accompanied by Gordon Lessells (Deputy Leader, UL), Bernd Kreussler (Team Leader, MIC Limerick), and Andrew Smith (Observer A, UCD).

1 Team selection and preparation

The team detailed in Table 1 consisted of those six students (in order) who scored highest in the Irish Mathematical Olympiad (IrMO), which was held for the 32nd time on Saturday, 11th May, 2019. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The students who participated in the IrMO sat the exam simultaneously in one of five *Mathematics Enrichment Centres* (UCC, UCD, NUIG, UL and MU). This year, a total of 86 students took part in the IrMO, 32 of whom were girls. The top performer is awarded the *Fergus Gaines cup*; congratulations to Lucas Bachmann, who achieved this honour in IrMO 2019.

The students who participate in the IrMO usually attend extra-curricular Mathematics Enrichment classes, which are offered at the five Mathematics Enrichment Centres listed in the previous paragraph. These classes run each year from January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions. More information on the organisation of these classes, as well as links to the individual maths enrichment centres, can be found at the Irish Maths Enrichment/IrMO website `http://www.irmo.ie/`.

Name	School	Year
Lucas Bachmann	Glenstal Abbey School, Murroe, Co. Limerick	6th
Tianyiwa Xie	Alexandra College, Milltown Road, Dublin 6	5th
Linhong Chen	The Institute of Education, Lower Leeson St, Dublin 2	5th
Alex Hanley	Lucan Community College, Lucan, Co. Dublin	5th
Laura Cosgrave	Midleton College, Midleton, Co. Cork	5th
Yunjie Wang	Christian Brothers College, Wellington Road, Cork	5th

Table 1: The Irish contestants at the 60th IMO

The selection and training for IMO 2019 followed procedures which are by now well-established. First, an Irish Maths Olympiad "Squad" was identified, consisting of the top performers in IrMO 2018 who were eligible to qualify for the Irish IMO team in 2019. For these students, a number of training camps was organised. Such training camps are very important, as during these mathematically intense 3–5 day events, students have the opportunity to socialise with their peers, exchange their mathematical ideas, and increase their motivation for their work throughout the year. A kick-start training camp, organised by Anca Mustata, was held in UCC from 22–25 August 2018. Classes were conducted by Eugene Gath, Conall Kelly, Claus Koestler, Bernd Kreussler, Declan Manning, Anca, Andrei and Anna Mustata, Cillian O'Doherty and Steve Wills.

Between the end of the UCC kick-start training camp and the beginning of the 2019 Mathematics Enrichment classes, the members of the Irish IMO Squad participated in the "remote training" programme, which operates as follows. At the beginning of each month from September to December inclusive, two sets of three problems are emailed to the participating students. They return their solutions or attempts by email to the proposer of the problems before the end of the month. The problem proposer then provides feedback on their work, as well as full solutions. This programme is very important for the successful engagement of "returning" students, and helps to develop the students' independence in mathematical problem solving. In 2018, 24 students comprised the Irish IMO Squad. The eight trainers involved in the remote training were Mark Flanagan, Eugene Gath, John Murray, Anca Mustata, Andrei Mustata, Prasanna Ramakrishnan, Harun Siljak and Andrew Smith.

Each year in November, the Irish Mathematical Olympiad starts with *IrMO Round 1*, a contest that is held in schools during a regular class period. In 2018, more than $14,000$ students, mostly in their senior cycle, participated in Round 1. Teachers were encouraged to hand out invitations to their best performing students to attend the mathematics enrichment classes in their nearest mathematics enrichment centre.

Having participated in other Mathematical Olympiads before is an advantage for students when they participate in the IMO. In the past six years, opportunities to do so have been created for members of the Irish Maths Olympiad Squad. In the current year, in addition to the possibility to compete in the European Girls' Mathematical Olympiad (EGMO), which is for female students only, the members of the Irish Maths Olympiad Squad were invited to participate in the Iranian Geometry Olympiad (6 September 2018), as well as the British Mathematical Olympiad Round 1 (30 November 2018) and Round 2 (24 January 2019). The exams in these three Olympiads are sat by the students at one of the five Enrichment Centres; no travel abroad was necessary. Thanks to the organisers of the IGO and to the UKMT, and in particular Geoff Smith, for giving our students these opportunities.

For all students who participate in enrichment classes, not only for the Squad members, each of the five maths enrichment centres hosts a local contest for the students, which takes place in February or March (each local contest is specific to its enrichment centre).

A number of training camps were organised in advance of IMO 2019. For the 2019/2020 Squad, to which five of the six members of the Irish team for IMO 2019 belong, a camp was held at Mary Immaculate College, Limerick, from 5–7 June 2019. At this camp, students gained exam experience in a $3\frac{1}{2}$ hours IMO-style exam in which they had to solve 3 problems. Two further training camps for the members of the Irish IMO team, featuring similar practice exams, were held in Limerick: at Mary Immaculate College from 2–4 July 2019 and at the University of Limerick from 9–12 July 2019. The second of these camps was

a joint camp with the IMO team from Trinidad and Tobago. The camps were organised by Bernd Kreussler and Gordon Lessells. The sessions at these camps were conducted by Mark Burke, Mark Flanagan, Ronan Flatley, Eugene Gath, Bernd Kreussler, Jim Leahy, Gordon Lessells, Anca Mustata, Andrei Mustata, Anna Mustata, Jagdesh Ramnanan and Andrew Smith.

2 The days in Bath

The Jury of the 60$^{\text{th}}$ IMO gathered at the Celtic Manor Resort near Newport in South Wales. The Team Leader and the Observer A travelled to the Celtic Manor on Thursday, 11 July. The Jury, which is composed of the Team Leaders of the participating countries and a Chairperson who is appointed by the organisers, is the prime decision making body for all IMO matters. Its most important task is choosing the six contest problems out of a shortlist of 32 problems provided by the IMO Problem Selection Committee, also appointed by the host country.

This year's Chairperson of the IMO Jury was Prof. Adam McBride, who carried out the same role also in 2002, the last time the IMO took place in the United Kingdom. He chaired the Jury meetings in a gentle yet very efficient manner, without sitting down for a single minute.

During the first Jury meetings, Leaders articulated their first impressions about the merits and beauty of all the shortlisted problems. Four of the easier problems needed to be removed from the shortlist, because they were too similar to problems that were used in other competitions in the past. Like in recent years, the Leaders felt that there was a lack of sufficiently many suitable easy problems on the shortlist. An appeal was made to submit more problems that may fit in this category for future IMOs. After intense discussion and debate, in the early afternoon of Saturday, the six problems for this year's IMO paper were selected.

It was the seventh year in a row that a problem selection protocol was followed whereby one problem from each of the four areas (algebra, combinatorics, geometry and number theory) would be included in problems 1, 2, 4 and 5. This protocol has the principal advantage of ensuring a balance between the four areas among the less difficult problems in the contest. As is now standard procedure at the IMO, an electronic voting mechanism was used during the Jury meetings, ensuring both efficiency and anonymity in voting procedures.

The opening ceremony of IMO 2019 took place on the 15$^{\text{th}}$ of July in The Forum in Bath, a former cinema built in the 1930s, which is now used as a venue for concerts and other large events as it has the largest seating capacity in Bath. There were two very short speeches, one by Geoff Smith, the President of the IMO Advisory Board, and the other by the Deputy Director of the UK Mathematics Trust, Stephen O'Hagan. The main part of the opening ceremony consisted of the parade of the teams. The Master of Ceremony of this less-than-one-hour event was a DJ who tried to entertain the contestants.

The two IMO contest exams took place on the 16$^{\text{th}}$ and 17$^{\text{th}}$ of July, starting at 8:30 each morning. During the first 30 minutes of the exams each day, students can ask questions regarding the IMO paper. Such questions can resolve ambiguities and ensure that students understand clearly the formulation of any contest problem. The answers are composed to resolve these difficulties, without providing any hint as to how to solve the problem. The questions of the students were scanned and sent to the leaders' site, from where the answers

are returned in the same way. This year's Q&A sessions were very efficient. On each day, 34 students had questions and these were answered by 9:15, this is only 45 minutes after the start of the contest.

The students' scripts from Day 1 became available at 8pm on the evening of the first day of the contest. After an initial brief study of the scripts it seemed that Lucas had solved all three problems on Day 1, an exciting surprise. Also, Tianyiwa's and Alex's solutions to Problem 1 looked promising. On Day 2, Andrew, Gordon and myself met with the team directly after the contest. Thereafter we began a detailed study of the scripts of the second day.

The final marks for each contestant are agreed in a process known as coordination. This important part of the IMO is well-established and ensures that the scripts of the students from so many different nations are marked fairly and consistently. The decisions in this process are based on very detailed and strict marking schemes which were prepared by the coordination teams, presented and defended to the Jury by the problem captains and agreed by the Jury – in some cases after significant changes to the first draft.

The marking of the scripts of each participating country is undertaken by two independent groups. One group consists of the Team Leader, the Deputy Leader and the Official Observer. The second group consists of the coordinators, who were appointed by the local organisers. This year's coordination schedule for our team was particularly tight: the half-hour meetings with the coordinators for problems 1, 2, 4 and 5 were scheduled to take place on the day immediately after the contest.

The help of Andrew as Observer A was essential to get us prepared in time for the coordination sessions. In total we had to study 227 pages of solutions and rough work of our contestants. In preparation for the coordination meetings, we needed to have a full understanding of the solution or attempts of each of our six students so that we could explain the merits of the students' work to the coordinators.

Each coordinator works on one problem only, but has to look at the solutions of the students from almost 20 teams, more than 100 students. Even though we had more time than the coordinators per problem and student, in most cases we came to the same conclusion regarding the points to be awarded. Due to the leadership of the chief coordinator, Imre Leader, and the professionalism of all coordinators, the coordination process went very smoothly at this year's IMO.

Problem 1, a functional equation, was solved in an efficient and straightforward manner by Lucas. Tianyiwa's solution was more complicated to understand. To agree on seven points with the coordinators, we needed to explain her writings in detail. We were able to do so successfully in the short time available to us thanks to the presence of an Irish Observer A. Alex's work on this problem fell short of a full solution because he did not make the connection between a usefully simplified form of the functional equation and the linearity of the solution function. This was a narrowly missed Honourable Mention.

Problem 4, a number theory problem, was more difficult than usual. Laura was the only member of the Irish team who solved this problem. Reading through the 22 pages she submitted for this problem was enjoyable, because she combined in a clever way estimates coming from the 2-adic and the 5-adic valuations of both sides of the equation. As a result, she only needed to check a few cases for small k by direct calculation. We needed to point the coordinators to a particular detail in Laura's solution to save her full marks for this question. Again, the availability of an Irish Observer A made it possible for us to enter the coordination meeting with such a detailed preparation.

During the two days of coordination, excursions and other activities were organised for

the students. Our students enjoyed mostly the trip to Oxford on Friday, 19th July, where Andrew Wiles gave a lecture to them.

The final Jury meeting, at which the medal cut-offs were decided, took place on Friday evening. At this meeting, the chief invigilator, Jeremy King, reported an irregularity: at the end of the exam at 1pm on Day 1, two students continued to write their solutions even after repeatedly being told to stop doing so. The Jury decided to set the score for their best question on Day 1 to zero.

The closing ceremony was held on Sunday, 21st July, followed by a Fun Fair and a Farewell Banquet with live music that evening. The team returned to Ireland on 22nd July.

3 The problems

The two exams took place on the 16th and 17th of July, starting at 8:30 each morning. On each day, $4\frac{1}{2}$ hours were available to solve three problems.

First Day

Problem 1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

(South Africa)

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

(Ukraine)

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

Second Day

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

(El Salvador)

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

(USA)

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

(India)

4 The results

The Jury tries to choose the problems such that Problems 1 and 4 are the most accessible, while Problems 2 and 5 are more challenging. Problems 3 and 6 are usually the most difficult problems, whose existence on the paper is justified in posing a sizeable challenge even to the top students in the IMO competition. Table 2, which shows the scores achieved by all contestants on the 6 problems, illustrates that this gradient of difficulty was generally maintained this year also. However, comparing average scores it can be said that Problem 4 was slightly harder and Problem 5 much easier than problems 4 and 5 have been in the past decade.

The medal cut-offs were as follows: 31 points needed for a Gold medal (52 students), 24 for Silver (94 students) and 17 for Bronze (156 students). A further 144 students were awarded an Honourable Mention (an Honourable Mention is awarded to any student who did not win a medal, but achieved 7 points out of 7 on at least one problem). Overall, 37.8 % of the possible points were scored by the contestants, which is one percent more than last year. A higher percentage of the possible points was achieved only at two IMOs in the past 20 years, in 2004 and 2014.

	P1	P2	P3	P4	P5	P6
0	73	251	520	211	156	558
1	65	135	46	63	20	25
2	6	30	3	4	168	7
3	24	6	6	7	12	0
4	14	6	5	13	5	1
5	5	3	9	19	7	0
6	52	92	4	47	3	3
7	382	98	28	257	250	27
average	5.179	2.399	0.572	3.736	3.567	0.403

Table 2: The number of contestants achieving each possible number of points on Problems 1–6

Name	P1	P2	P3	P4	P5	P6	total	relative ranking	award
Lucas Bachmann	7	7	7	1	7	0	29	91.29 %	Silver Medal
Tianyiwa Xie	7	1	0	1	2	0	11	37.90 %	Hon. Mention
Laura Cosgrave	1	0	0	7	0	0	8	28.55 %	Hon. Mention
Alex Hanley	4	0	0	1	2	0	7	25.48 %	
Linhong Chen	1	2	0	0	2	0	5	22.90 %	
Yunjie Wang	1	0	0	0	0	0	1	13.06 %	

Table 3: The results of the Irish contestants

Table 3 shows the results of the Irish contestants. The team scored a total of 61 points, the fifth best score of an Irish team at the IMO. Lucas Bachmann won a Silver medal, having completely solved four problems – this is a fantastic achievement. Also, Tianyiwa Xie and Laura Cosgrave won an Honourable Mention for their complete solutions to Problems 1 and 4, respectively.

The figures in Table 4 have the following meaning. The first figure after the topic indicates the percentage of all points scored out of the maximum possible. The second number is the same for the Irish team and the final column indicates the Irish average score as a percentage of the overall average. This year the relative performance of the Irish team on problems 1 and 4 was not as good as in the past three years.

Problem	topic	all countries	Ireland	relative
1	algebra	74.0	50.0	67.6
2	geometry	34.3	23.8	69.5
3	combinatorics	8.2	16.7	204.1
4	number theory	53.4	23.8	44.6
5	combinatorics	51.0	31.0	60.7
6	geometry	5.8	0.0	0.0
all		37.8	24.2	64.1

Table 4: Relative results of the Irish team for each problem

It is also worth mentioning here that some young Irish mathematicians won awards this year in Mathematical Olympiads other than IMO. At the European Girls' Mathematical Olympiad (EGMO) 2019 in Kyiv, Ukraine, Tianyiwa Xie won a Bronze Medal and Laura

Cosgrave and Yixin Huang won Honourable Mentions. In addition, Lucas Bachmann won a Bronze medal at the 5$^{\text{th}}$ Iranian Geometry Olympiad (IGO) in September 2018.

This year, six students achieved a "perfect score" (42 points) at the IMO. Although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were China and the USA (both 227 points) closely followed by South Korea (226 points). Ireland, with 61 points in total, came in 71$^{\text{st}}$ place, which corresponds to a relative ranking of 36.94%. This is the fourth best relative ranking an Irish team achieved since the start of its involvement with the IMO in 1988. Three of the four top relative rankings of the Irish team were achieved within the past 6 years.

The detailed results can be found on the official IMO website, which is located at `http://www.imo-official.org`.

5 Outlook

The next countries to host the IMO will be

2020	Russian Federation	8–18 July
2021	United States of America	7–16 July
2022	Norway	6–16 July
2023	Japan	2–13 July

6 Conclusions

The outstanding result of this year's IMO, from an Irish perspective, is Lucas Bachmann's Silver Medal. This is the second Silver Medal ever achieved by an Irish student – the first one was won by Fiachra Knox in 2005. Lucas' success was reported in newspapers such as the Irish Times, the Limerick Leader and the Limerick Post.

This is the third consecutive year in which the Irish team came home with at least one medal, this has not happened before in the history of Irish participation in the IMO. More than half of all Honourable Mentions achieved by Irish IMO contestants since Ireland's first IMO participation in 1988 were achieved within the last seven years. This is evidence that while there are fluctuations in performance year on year, a generally sustained team-level improvement can be detected within the last few years. The extra effort being invested in training activities in the last few years shows a clear correlation with this improvement.

It is of primary importance that sufficient funding becomes available for the activities detailed above, in particular for the training camps. An increased level of funding would also allow the scope of these initiatives to be widened further, so that the performance of Irish students in international mathematics contests can continue to improve year on year.

It is interesting to note that the four top relative rankings of the Irish team at the IMO were achieved either when all six team members got an award (2014, 2017), or one team member received a Silver Medal (2005, 2019). One obvious conclusion from this observation is that to improve the performance of the Irish team we need to increase efforts to enable a larger number of potential Irish contestants to perform at an internationally competitive level.

In recent years, initiatives have been started all over Ireland that aim at involving Junior Cycle students in problem-solving activities. The most noteworthy are Junior Maths

Enrichment programmes, the PRISM (Problem Solving for Post-Primary Schools) competition and the Maths Circles initiative. Some of these activities are externally funded and their continuation depends on the availability of future funding. It seems essential for the long-term improvement of Irish teams at the IMO that the problem-solving activities offered for younger students are maintained or even extended, because students who become involved in problem-solving activities at an earlier age have a much enhanced probability to reach an internationally competitive level.

The sending of a full team of six students, together with Leader, Deputy Leader and Observer, to the IMO contest requires sustained funding. It would be very beneficial for the team leadership at future IMOs if the practice of sending an Irish Observer to the IMO could be continued in subsequent years.

7 Acknowledgements

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The principal foundation for the success of the contestants is the work done with the students in the Maths Enrichment Programmes at the five universities. This work is carried out for free by volunteers in their spare time. Thanks go to this year's trainers at the five Irish centres:

- At UCC:

  Tom Carroll, Michael Cronin, David Goulding, David Henry, Philipp Hoevel, Conall Kelly, Arundhathi Krishnan, Declan Manning, Anca Mustata, Andrei Mustata, Jacob Bennett Woolf.

- At UCD:

  Kazim Buyukboduk, Anthony Cronin, Fabio Deelan Cunden, Mark Flanagan, Marius Ghergu, Mary Hanley, Kevin Hutchinson, Samuel Johnston, Thomas Laffey, Myrto Manolaki, Gary McGuire, Harun Siljak, Helena Smigoc, Andrew Smith.

- At NUIG:

  John Burns, Angela Carnevale, Graham Ellis, Niall Madden, Goetz Pfeiffer, Kirsten Pfeiffer, Rachel Quinlan, Tobias Rossmann, Nina Snigireva.

- At UL:

Mark Burke, Ronan Flatley, Mary Frawley, Sarah Frawley, Eugene Gath, Bernd Kreussler, Jim Leahy and Gordon Lessells.

- At MU:

  Stefan Bechtluft-Sachs, Jacquie Birkett, Stephen Buckley, Peter Clifford, Rafael de Andrade Moral, Liam Jordon, David Malone, Ollie Mason, Jack Mc Donnell, John Murray, Anthony O'Farrell, Adam Ralph, David Redmond, Mark Walsh.

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