Ireland's Participation in the 58th International Mathematical Olympiad

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The 58$^{\text{th}}$ International Mathematical Olympiad (IMO) took place in Rio de Janeiro, Brazil, from 12–23 July 2017. A total of 615 students (62 of whom were girls) participated from 111 countries. This is the largest number of participating countries in IMO history. This year also saw the first ever IMO participation of Nepal.

The Irish delegation consisted of six students (see Table 1), the Team Leader, Mark Flanagan (UCD) and the Deputy Leader, Anca Mustaţa (UCC).

1 Team selection and preparation

The team detailed in Table 1 consisted of those six students (in order) who scored highest in the Irish Mathematical Olympiad (IrMO), which was held for the 30$^{\text{th}}$ time on Saturday, 6$^{\text{th}}$ May, 2017. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The students who participated in the IrMO sat the exam simultaneously in one of five *Mathematics Enrichment Centres* (UCC, UCD, NUIG, UL and MU). This year, a total of 91 students took part in the IrMO. The top performer is awarded the *Fergus Gaines cup*; congratulations to Cillian Doherty, who achieved this honour in IrMO 2017.

The students who participate in the IrMO usually attend extra-curricular Mathematics Enrichment classes, which are offered at the five Mathematics Enrichment Centres listed in the previous paragraph. These classes run each year from January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions. More information on the organisation of these classes, as well as links to the individual maths enrichment centres, can be found at the Irish Maths Enrichment/IrMO website `http://www.irmo.ie/`.

Name	School	Year
Cillian Doherty	Coláiste Eoin, Booterstown, Co. Dublin	6$^{\text{th}}$
Antonia Huang	Mount Anville Secondary School, Dublin 14	5$^{\text{th}}$
Mark Heavey	Blackrock College, Blackrock, Co. Dublin	6$^{\text{th}}$
Anna Mustaţa	Bishopstown Community School, Cork	5$^{\text{th}}$
Mark Fortune	CBS Thurles Secondary School, Thurles, Co. Tipperary	6$^{\text{th}}$
Darragh Glynn	St. Paul's College, Raheny, Dublin 5	6$^{\text{th}}$

Table 1: The Irish contestants at the 58$^{\text{th}}$ IMO

The selection and training for the IrMO 2017 contest followed procedures which are by now well-established. First, an Irish Maths Olympiad "Squad" was identified, consisting of the top performers in IrMO 2016 who were eligible to qualify for the Irish IMO team in 2017. For these students, a kick-start training camp was held in UCC from 17–20 August 2016; such training camps are very important, as during these mathematically intense 3–5 day events, students have the opportunity to socialise with their peers, exchange their mathematical ideas, and increase their motivation for their work throughout the year.

Between the end of the UCC kick-start training camp and the beginning of the 2017 Mathematics Enrichment classes, the members of the Irish IMO Squad participated in the "remote training" programme, which operates as follows. At the beginning of each month from September to December inclusive, two sets of three problems are emailed to the participating students. They return their (complete or incomplete) solutions, by email or by post, to the proposer of the problems before the end of the month. The problem proposer then provides feedback on their work, as well as full solutions. This programme is very important for the successful engagement of "returning" students, and helps to develop the students' independence in mathematical problem solving. In 2016, 14 students comprised the Irish IMO Squad, and the eight trainers involved in the remote training were Mark Flanagan, Eugene Gath, Bernd Kreussler, Gordon Lessells, John Murray, Anca Mustaţa, Andrei Mustaţa and Rachel Quinlan.

Each year in November, the Irish Mathematical Olympiad starts with *IrMO Round 1*, a contest that is held in schools during a regular class period. In 2017, more than 14,000 students, mostly in their senior cycle, participated in Round 1. Teachers were encouraged to hand out invitations to their best performing students to attend the mathematics enrichment classes in their nearest mathematics enrichment centre.

Each of the five maths enrichment centres hosts a local contest for the students, which takes place in February or March (each local contest is specific to its enrichment centre). In addition, this year a number of students from Ireland was invited to participate in the British Mathematical Olympiad Round 1 (2 December 2016) and Round 2 (26 January 2017). This is a great opportunity for talented students as they get to experience challenging problem solving in a real olympiad-style environment. Thanks to UKMT, and in particular Geoff Smith, for giving our students this opportunity.

Two further training camps were also organised at various locations shortly in advance of IrMO 2017. A training camp for the top performing students in IrMO 2017 was held at Mary Immaculate College, Limerick, from 7–9 June 2017, featuring an IMO-style exam in which $3\frac{1}{2}$ hours were given to solve 3 problems. A training camp for the six members of the Irish IMO team was held at University College Cork from 3–7 July 2017. The camps were organised by Bernd Kreussler and Anca Mustaţa.

A final joint training camp was held immediately before the IMO. This camp was held at the Windsor Florida Hotel in Rio de Janeiro. The sessions were conducted by Anca Mustaţa and Mark Flanagan.

2 The days in Rio de Janeiro

The Irish IMO team, together with the Leader and Deputy Leader, arrived in Rio de Janeiro on the evening of Monday the 10^{th} of July. We proceeded to the Windsor Florida Hotel, at which the team would carry out their intensive pre-IMO training camp. We were glad to discover that the hotel staff were extremely accommodating, allowing us free use of the spacious top floor of the hotel for our training sessions. Led by the Irish Team Leader and Deputy Leader, these sessions consisted of many hours of intensive problem-solving. This period also allowed our students to acclimatise to the warm weather and the time difference. On July 13, I left Anca and the students at the training camp and travelled to the Jury site.

The Jury of the IMO, which is composed of the Team Leaders of the participating countries and a Chairperson who is appointed by the organisers, is the prime decision making body for all IMO matters. Its most important task is choosing the six contest problems out of a shortlist of 32 problems provided by the IMO Problem Selection Committee, also appointed by the host country. This year's Chairperson of the Jury was Dr. Nicolau Saldanha. Nicolau's inimitable style made the Jury meetings both very pleasant and very efficient.

The Jury meetings involved much intense discussion and debate around choosing the 6 problems for the IMO paper. As in recent years, a problem selection protocol was followed whereby one problem from each of the four areas (algebra, combinatorics, geometry and number theory) would be included in problems 1, 2, 4 and 5. This protocol has the principal advantage of ensuring a balance between the four areas among the less difficult problems in the contest.

Apart from the usual business of the Jury, some specific points were notable this year. For the second year running, electronic voting machines were used during the Jury meetings. This worked very well, lending both efficiency and anonymity to voting procedures. Also, significant changes were made to the procedures surrounding Q&A this year. The term "Q&A" refers to the period of the first 30 minutes of the IMO contest each day, wherein students can ask questions regarding the IMO paper. Such questions can resolve ambiguities and ensure that students understand clearly the formulation of any contest problem (the answers are composed to resolve these difficulties, without providing any hint as to how to solve the problem). This year, a quite new procedure was introduced regarding Q&A; instead of questions being answered one-by-one by the Jury, a new system was devised by means of which the students' questions could be processed in parallel. The new procedure was very well-organised, and led to an notably efficient Q&A session.

On the 15^{th} of July, the jury meeting was interrupted to announce the event of the death of Maryam Mirzakani, a female mathematician who participated in IMO 1994 and 1995, obtaining a perfect score in IMO 1995. Maryam won the Fields Medal, the highest honor in Mathematics, in 2014. Her loss comes as a great blow to the mathematics community of the world.

The opening ceremony of IMO 2017, which took place on the 17^{th} of July, was very lively and upbeat, featuring singers, clowns and an excellent samba band. The informal atmosphere of the ceremony helped to relax the contestants before their challenging contest. The two exams took place on the 18^{th} and 19^{th} of July, starting at 9 o'clock each morning.

The students' scripts from Day 1 became available on the evening of the first day of the contest. On my initial study of the scripts it became clear that the students had performed very well, providing some very nice solutions to some difficult problems. On Day 2, myself and Anca met with the team directly after the contest, and thereafter we began a detailed study of the scripts of the second day. These scripts showed us that the students had managed to sustain their high level of performance from Day 1.

Problem 1 was more accessible than usual for IMO, but still required mathematical professionalism to solve and logical rigour in writing up the solution. The Irish students rose to this challenge, scoring a total of 36 out of 42 possible points on this problem. Problem 4, which was a geometry problem, was more difficult; however, here the Irish students provided some interesting and varied solutions; many of the Irish students used the key idea of completing a parallelogram; in addition, Antonia used the power of a point extensively, while Mark Fortune completed his solution by applying the concept of spiral similarity. Problem 2, a functional equation, was exceedingly difficult for a medium problem, and only a minority (25.7%) of students in the contest scored more than 3 points on this problem. One such student was Ireland's Cillian Doherty; not only did Cillian solve the first part of the problem, he also found an elegant way of showing that the solution can be completed if the function in the problem can be shown to be injective. Anna had excellent ideas on Problem 5, although a stringent marking scheme meant that more than 2 points on this problem were awarded only to students who had already found the key idea for the solution. Problems 3 and 6, the most difficult problems in the competition, had some attractive properties this year: Problem 3 had a very intriguing and accessible problem statement, while Problem 6, concerning homogeneous polynomials, had the nice property of possessing connections to research mathematics.

The final Jury meeting, at which the medal cut-offs were decided, took place on Tuesday 21$^{\text{st}}$ July. The closing ceremony was held on the following day, followed by a Farewell Party that evening. The team returned to Ireland on Thursday 23$^{\text{rd}}$ July.

3 The problems

The two exams took place on the 18$^{\text{th}}$ and 19$^{\text{th}}$ of July, starting at 9 o'clock each morning. On each day, $4\frac{1}{2}$ hours were available to solve three problems.

First Day

Problem 1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geqslant 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

(South Africa)

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

(Albania)

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

(Austria)

Second Day

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

(Luxembourg)

Problem 5. An integer $N \geqslant 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

Show that this is always possible.

(Russia)

Problem 6. An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers a_0, a_1, …, a_n such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

(United States of America)

4 The results

The Jury tries to choose the problems such that Problems 1 and 4 are the most accessible, while Problems 2 and 5 are more challenging. Problems 3 and 6 are usually the most difficult problems, whose existence on the paper is justified in posing a sizeable challenge even to the top students in the IMO competition. Table 2, which shows the scores achieved by all contestants on the 6 problems, illustrates that this gradient of difficulty was generally maintained this year also.

	P1	P2	P3	P4	P5	P6
0	40	183	608	47	451	557
1	16	110	3	93	46	24
2	17	26	0	42	47	9
3	5	138	0	14	9	5
4	12	79	1	15	0	4
5	54	10	1	4	2	2
6	25	8	0	6	1	0
7	446	61	2	394	59	14
average	5.943	2.304	0.042	5.029	0.969	0.294

Table 2: The number of contestants achieving each possible number of points on Problems 1–6

The medal cut-offs were as follows: 25 points needed for a Gold medal (48 students), 19 for Silver (90 students) and 16 for Bronze (153 students). A further 222 students were awarded an Honourable Mention (an Honourable Mention is awarded to any student who did not win a medal, but achieved 7 points out of 7 on at least one problem). Overall, only 34.7 % of the possible points were scored by the contestants, compared to the figure of 35.2 % last year. This low fraction of points scored by the contestants shows that, similarly to last year, this was a very difficult IMO.

Table 3 shows the results of the Irish contestants. The total team score this year (80 points) was the highest ever achieved by an Irish team in Irish IMO participation history. Also, apart from this very strong team performance, Cillian Doherty and Anna Mustata both won Bronze medals. This is the first time ever that an Irish team has brought home more than one medal at an IMO. All four of the other team members won Honourable Mentions; this represents an outstanding overall team achievement.

The figures in Table 4 have the following meaning. The first figure after the problem number indicates the percentage of all points scored out of the maximum

Name	P1	P2	P3	P4	P5	P6	total	relative ranking	award
Cillian Doherty	7	4	0	6	0	0	17	69.54%	Bronze Medal
Anna Mustata	7	0	0	7	2	0	16	57.00%	Bronze Medal
Antonia Huang	7	0	0	7	0	0	14	44.46%	Hon. Mention
Mark Heavey	7	3	0	4	0	0	14	44.46%	Hon. Mention
Darragh Glynn	7	1	0	3	0	0	11	28.18%	Hon. Mention
Mark Fortune	1	0	0	7	0	0	8	19.22%	Hon. Mention

Table 3: The results of the Irish contestants

Problem	topic	all countries	Ireland	relative
1	number theory	84.9	85.7	101.0
2	algebra	32.9	19.0	57.9
3	combinatorics	0.6	0.0	0.0
4	geometry	71.8	81.0	112.7
5	combinatorics	13.8	4.8	34.4
6	number theory	4.2	0.0	0.0
all		34.7	31.7	91.4

Table 4: Relative results of the Irish team for each problem

possible. The second number is the same for the Irish team and the last column indicates the Irish average score as a percentage of the overall average.

It can be clearly seen that the Irish students' performance was at an internationally competitive level on Problem 1 (number theory), while the Irish team performed above the international average on Problem 4 (geometry), as given in Table 4. This shows the continuation of a sustained improvement over the last few years; indeed, the Irish team's performance was seen to be approaching the international average on these same mathematical topics last year.

It is noteworthy that two of the Irish contestants this year won awards in Mathematical Olympiads other than IMO. At the European Girls' Mathematical Olympiad (EGMO) 2017 in Zürich, Switzerland, Anna Mustata won a Silver medal, and Antonia Huang won a Bronze medal. In addition, Antonia Huang won a Bronze medal at the Iranian Geometry Olympiad (IGO). Congratulations to Anna and to Antonia on these great achievements.

This year, the top individual score was 35 points, and was achieved by three students hailing from Iran, Japan and Vietnam. Also, although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were the Republic of Korea (170 points), China (159 points) and Vietnam (155 points). Ireland, with 80 points in total, shared the 62$^{\text{nd}}$ place with Belgium and Sri Lanka.

The detailed results can be found on the official IMO website, which is located at `http://www.imo-official.org`.

5 Outlook

The next countries to host the IMO will be

2018	Romania	3–14 July
2019	United Kingdom	11–22 July
2020	Russian Federation	
2021	United States of America	
2022	Norway	

6 Conclusions

This year, the Irish IMO team obtained the highest team score ever achieved by an Irish team in the history of Irish IMO participation. Two of the Irish team members, Cillian Doherty and Anna Mustata, won bronze medals. All four of the other team members won Honourable Mentions. This is a truly excellent result; since Ireland's first participation in 1988, the Irish teams have won 10 medals in total. This historic performance caught the attention of mainstream media, being reported by RTE and Silicon Republic, among others.

Also, of the 41 Honourable Mentions achieved by Irish IMO contestants since Ireland's first IMO participation in 1988, 22 were achieved within the last six years. This is evidence that while there are fluctuations in performance year on year, a generally sustained team-level improvement can be detected within the last few years. The extra effort being invested in training activities in the last few years shows a clear correlation with this improvement. However, it is of course important to maintain as well as to build upon this improved performance in the longer term.

It is noteworthy that students who become involved in problem-solving activities at an earlier age have a much enhanced probability to succeed at a high level. Therefore, it is important that students are engaged in problem-solving at an early age. Some activities in this direction have recently been very successful. A Junior Maths Enrichment programme, consisting of mathematical problem-solving activities for Junior Cycle students, has been running in the maths enrichment centre at UCC for four years and is by now well-established; this initiative is a by-product of the "Maths Circles" initiative which was set up for Junior Cycle students in second level schools in the Cork area in 2013. Apart from this established programme in Cork, in 2017 Junior Maths Enrichment activities were also rolled out for the first time in Limerick and Galway, with more than 350 students participating collectively over the three centres. Feedback from parents and teachers suggests that the demand for such activities is huge and that the participants derived great enjoyment from these classes. It would be extremely good if such early-stage regional activities became more widespread, and if teachers can be motivated to support problem-solving activities at a local level.

It is worth also mentioning in this context the PRISM (Problem Solving for Post-Primary Schools) competition, which is a multiple-choice mathematical problem solving contest offered both for students in Junior Cycle and for students in Senior Cycle. This contest is organised since 2006 by mathematicians from NUI Galway, and usually takes place in October every year.

Students who achieve excellent results at the IMO are invariably students who immerse themselves in mathematical problem-solving activities. Therefore, students must be given the opportunities and the supports required for them to develop the skills to work intensively on problems in their own time, and without the necessity of the time-intensive guidance of a trainer. While our current training activities are very beneficial to students in that they provide an entry point to mathematical problem-solving activities, in order to be successful students must reach a level of independence where they can work on their own. It is clear that the remote training programme for the Irish IMO Squad is helping Irish students to develop a high level of independence in problem-solving, while maintaining a structured form of support from trainers. The current challenge is how to involve more students in this remote training programme, as well as how to nurture such problem-solving independence among the wider group of participants in the national Mathematics Enrichment Programme.

Of course, these initiatives cannot succeed without the requisite financial support. The delivery of the initiatives described above, as well as the running of training camps and the sending of a full team of six students, together with Leader and Deputy Leader, to the IMO contest requires sustained funding. It is of primary importance that sufficient funding becomes available for these activities. An increased level of funding would also allow the scope of these initiatives to be widened further, so that the performance of Irish students in international Mathematics contests can continue to improve year on year. Additional funding would also allow the reinstatement of the practice of sending an Irish Observer to the IMO; this was found to be a very beneficial practice in the past, however due to funding limitations this has not been possible in recent years.

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- At UL:

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