8. Lemoine Point and Circles.

Theorem 1 *In a triangle ABC the distances from the Lemoine point L to the sides are in the ratio*

$$\alpha a, \alpha b, \alpha c,$$

where $\alpha = \dfrac{2\, area(ABC)}{a^2+b^2+c^2}$ *and a, b, c denote the lengths of the sides BC, CA and AB, respectively (Figure 1).*

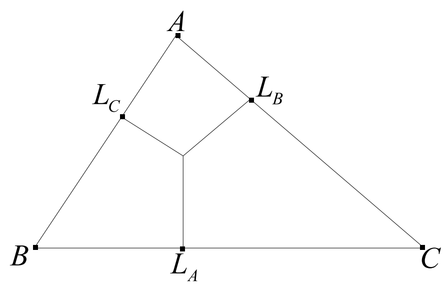

Figure 1:

Proof By Grebe's theorem (Theorem 6 of Set 7), if L_A, L_B and L_C are the feet of perpendiculars from the Lemoine point L to the sides BC, CA and AB, we have

$$\frac{|LL_A|}{a} = \frac{|LL_B|}{b} = \frac{|LL_C|}{c} = \alpha, \text{ say.}$$

Thus $|LL_A| = \alpha a, |LL_B| = \alpha b$ and $|LL_C| = \alpha c$.

Furthermore, if $S = area(ABC)$, then

$$\begin{aligned} 2S &= 2\,area(LBC) + 2\,area(LCA) + 2\,area(LAB) \\ &= a|LL_A| + b|LL_B| + c|LL_C|. \end{aligned}$$

Thus

$$\begin{aligned} 2S &= a(\alpha a) + b(\alpha b) + c(\alpha c) \\ &= \alpha(a^2 + b^2 + c^2) \end{aligned}$$

giving $\alpha = \dfrac{2S}{a^2 + b^2 + c^2}$. Result follows.

Theorem 2 *The sides of the Lemoine's pedal triangle are*

$$2\alpha m_a, 2\alpha m_b \text{ and } 2\alpha m_c,$$

where m_a, m_b and m_c are the lengths of the medians from the vertices A, B and C respectively (Figure 2) and

$$\alpha = \frac{2\,area(ABC)}{a^2 + b^2 + c^2}.$$

Proof Since AL_BLL_C is a cyclic quadrilateral

$$\widehat{L_CLL_B} = \pi - \widehat{A}.$$

Applying the cosine rule to the side L_CL_B of the triangle LL_CL_B.

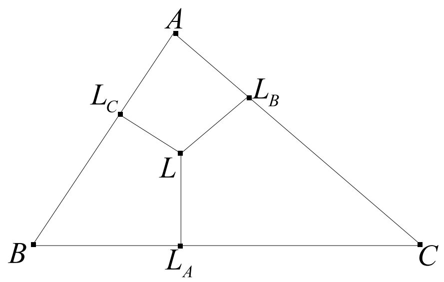

Figure 2:

$$
\begin{aligned}
|L_C L_B|^2 &= |LL_C|^2 + |LL_B|^2 - 2|LL_C|.|LL_B| \cos(\widehat{L_C L L_B}) \\
&= \alpha^2 c^2 + \alpha^2 b^2 - 2\alpha^2 bc \cos(\pi - \widehat{A}) \\
&= \alpha^2 (b^2 + c^2 + 2bc \cos(\widehat{A}) \\
&= \alpha^2 (b^2 + c^2 + 2bc(\frac{b^2 + c^2 - a^2}{2bc})) \\
&= \alpha^2 (2(b^2 + c^2) - a^2) \\
&= \alpha^2 (4m_a^2), \text{ by the median property}
\end{aligned}
$$

where m_a is the length of the median from the vertex A.

Thus $|L_C L_B| = 2\alpha m_a$, as required. Similarly show $|L_B L_A| = 2\alpha m_c$ and $|L_A L_C| = 2\alpha m_b$. □

Next we derive some inequalities involving the expansion $a^2 + b^2 + c^2$ and $S = area(ABC)$.

Let X, Y and Z be points of the sides BC, CA and AB of the triangle ABC (Figure 3). In set 7 we considered the function

$$f(X, Y, Z) = |XY|^2 + |YZ|^2 + |ZX|^2$$

and proved that this has a minimum from the Lemoine point L to the sides, i.e.

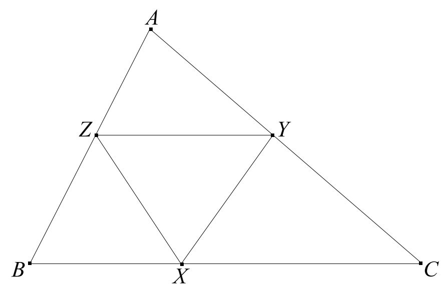

Figure 3:

$$
\begin{aligned}
f(X, Y, Z) &\geq f(L_A, L_B, L_C) \\
&= 4\alpha^2 (m_a^2 + m_b^2 + m_c^2), \quad \text{by theorem 2 above} \\
&= 4\alpha^2 (\frac{3}{4})(a^2 + b^2 + c^2), \quad \text{since } m_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4} \text{etc} \\
&= 3\alpha^2 (a^2 + b^2 + c^2) \\
&= \frac{12\, S^2}{a^2 + b^2 + c^2}.
\end{aligned}
$$

Now consider the following two particular cases.

Case 1. XYZ is the median triangle $A_1B_1C_1$ of the triangle ABC (Figure 4).

Then $|A_1B_1| = \frac{c}{2}, |B_1C_1| = \frac{a}{2}$ and $|C_1A_1| = \frac{b}{2}$. Thus

$$f(A_1, B_1, C_1) = |A_1B_1|^2 + |B_1C_1|^2 + |C_1A_1|^2 = \frac{1}{4}(a^2 + b^2 + c^2)$$

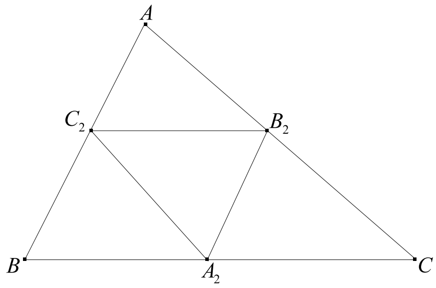

Figure 4:

and then, since $f(A_1, B_1, C_1) \geq f(L_A, L_B, L_C)$

$$\frac{1}{4}(a^2 + b^2 + c^2) \geq \frac{12\,S^2}{a^2 + b^2 + c^2}$$

which implies $a^2 + b^2 + c^2 \geq 4\sqrt{3}.S$.

Case 2. XYZ is the orthic triangle $A_2B_2C_2$ (Figure 5). Since $|B_2C_2| = a\cos(\widehat{A}), |C_2A_2| = b\cos(\widehat{B})$ and $|A_2B_2| = c\cos(\widehat{C})$ by Proposition 2 of Set 5, then

$$f(A_2, B_2, C_2) = a^2.\cos^2(\widehat{A}) + b^2.\cos^2(\widehat{B}) + c^2.\cos^2(\widehat{C})$$

and so $(a^2 + b^2 + c^2)(a^2.\cos^2(\widehat{A}) + b^2.\cos^2(\widehat{B}) + c^2.\cos^2(\widehat{C})) \geq 12\,S^2$.

Our next result gives us the area of the Lemoine pedal triangle.

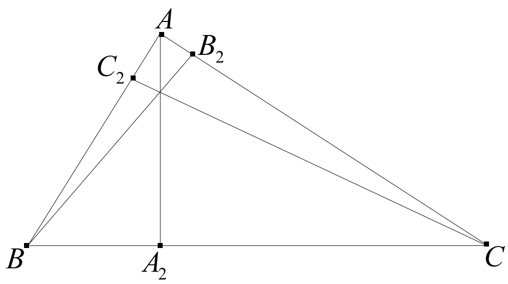

Figure 5:

Theorem 3 *The area of the Lemoine pedal triangle of a triangle ABC is given by*

$$\frac{12(area(ABC))^2}{(a^2 + b^2 + c^2)^2}$$

Proof Let A_1, B_1, C_1 be the midpoints of the sides BC, CA and AB of a triangle ABC (Figure 6). By theorem 3 above the Lemoine pedal triangle has sidelengths $2\alpha m_a, 2\alpha m_b$ and $2\alpha m_c$, where

$$\alpha = \frac{2\,S}{a^2 + b^2 + c^2}$$

and $m_a = |AA_1|, m_b = |BB_1|$ and $m_c = |CC_1|$.

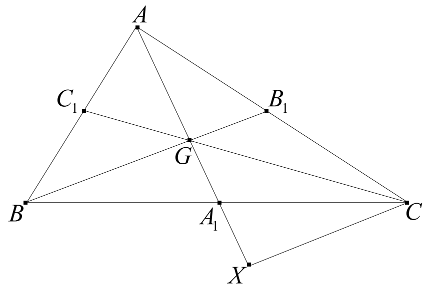

Figure 6:

In the triangle, G is the centroid and we extend the median AA_1 to a point X so that $|GA_1| = |A_1X|$.

Now consider the triangle GXC. We claim that the lengths of the sides are $\frac{2}{3}$ times the lengths of the three medians of the triangle ABC.

Clearly

$$\begin{aligned} |GC| &= \frac{2}{3}\,|CC_1| &= \frac{2}{3}\,m_c, and \\ |GX| &= 2|GA_1| &= 2(\frac{1}{3}|AA_1|) = \frac{2}{3}m_a. \end{aligned}$$

Finally, in the triangle AXC, the points G and B_1 are the midpoints of the sides AX and AC, respectively. Thus

$$|XC| = 2|GB_1| = 2(\frac{1}{3}|BB_1|) = \frac{2}{3}m_b.$$

This establishes the fact the claim about the lengths of the sides of the triangle GXC.

Next, let W be the area of a triangle with sides of length m_a, m_b and m_c. Then

$$area(L_A L_B L_C) = 4\alpha^2 W$$

and

$$area(GXC) = \frac{4}{9}\,W.$$

But

$$
\begin{aligned}
area(GXC) \quad &= 2\,area(GA_1C) \quad = 2(\frac{1}{6}\,area(ABC)) \\
&= \frac{S}{3}, \text{whereS} \quad = area(ABC) \\
&\text{Thus } \frac{S}{3} = \frac{4}{9}W \text{ so } W = \frac{3}{4}W
\end{aligned}
$$

Finally,

$$
\begin{aligned}
area(L_AL_BL_C) \quad &= 4\alpha^2 W \quad = 4\alpha^2(\frac{3}{4}\,S) \\
&= 3\alpha^2 S \quad = 3S.\frac{4\,S^2}{(a^2+b^2+c^2)^2} \\
&= \frac{12\,S^3}{(a^2+b^2+c^2)^2}.
\end{aligned}
$$

1 Lemoine Circles

Recall the following facts. Suppose X and Y are points on the sides AB and AC of a triangle ABC, then

(i) if XY is parallel to BC (Figure 7), the midpoint of XY lies on the median AA_1, and

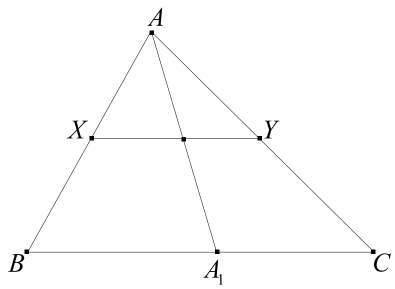

Figure 7:

(ii) if XY is antiparallel to BC (Figure 8), the midpoint of XY lies on the midpoint of the symmedian AA'_1.

Theorem 4 *(First Lemoine Circle). The antiparallels to the sides of a triangle passing through the Lemoine point generate six points on the sides which are concyclic.*

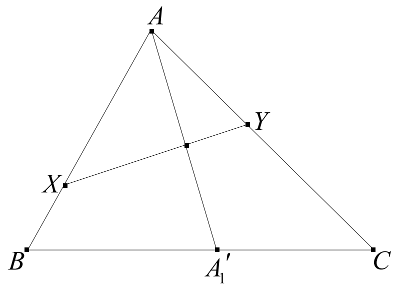

Figure 8:

Proof Let $B'C'$ be antiparallel to $BC, A''B''$ be antiparallel to AB and $A'''C'''$ be antiparallel to AC. The L (Lemoine point) lies on all the antiparallels.

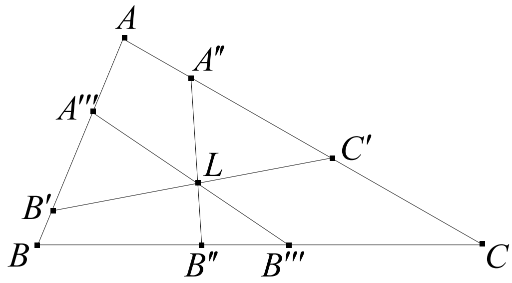

Figure 9:

The point L is the midpoint of $B'C'$ which is antiparallel to the side BC. Similarly L is the midpoint of the antiparallels $A''B''$ and $A'''C'''$. Next we claim that the triangle $LB'A'''$ is isosceles.

Since $B'C'$ is antiparallel to BC

$$\widehat{LB'A'''} = \widehat{C},$$

and since $C'''A'''$ is antiparallel to AC

$$\widehat{LA'''B'} = \widehat{C}$$

Thus

$$\widehat{LB'A'''} = \widehat{LA'''B'},$$

and so the triangle $LB'A'''$ is isoceles, as claimed. Thus $|LB'| = |LA'''|$. Since L is the midpoint of $B'C', A''B''$ and $A'''C'''$, it follows that

$$|LA'''| = |LB'| = |LB''| = |LB'''| = |LC'| = |LA''|.$$

Then the circle with L as centre and radius $|LA'''|$ passes through all six points. □

Theorem 5 *(Lemoine Second Circle)* *The parallels to the sides of a triangle passing through the Lemoine point generate six points on the sides which are concyclic.*

Proof Let $B'C'$ be parallel to $BC, B''A''$ parallel to AB and $A'''C'''$ be parallel to AC.

Considering the parallelogram $LA'''AA''$, the diagonals AL and $A'''A''$ bisect one another (Figure 10).

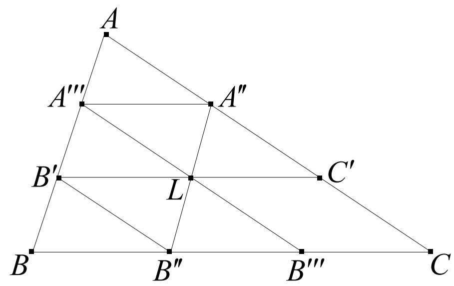

Figure 10:

Thus $A'''A''$ is antiparallel to AB and $B'B''$ is antiparallel to AC.

Next we claim that $A''B''B'A'''$ is a cyclic quarilateral.

$$\widehat{B''A''A'''} = \widehat{A''A'''A}, \text{sinceB}''\text{A}''\text{isparalleltoAB}$$
$$= \widehat{C}, \text{sinceA}'''\text{A}''\text{isantiparalleltoBC}$$

$$\widehat{B''B'A'''} = 180^\circ - \widehat{B''B'B}$$
$$= 180^\circ - \widehat{C}, \text{sinceB}'\text{B}''\text{isantiparallel toAC}$$
$$= 180^\circ - ?\widehat{B''?A''A'''}$$

If follows that $A''B''B'A'''$ is a cyclic quarilateral. Similarly it can be shown that $A'''A''C'C'''$ is a cyclic quarilateral.

Since $A'''A''$ is antiparallel to BC and BC is parallel to $B'C'$. Thus $A'''A''C'B'$ is a cyclic quarilateral. Thus

$$B' \in \mathcal{C}(A'''A''C'), \text{ the circumcircle of } A'''A''C'$$

Since $A'''A''C'C'''$ is cyclic, the point C''' also belongs to $\mathcal{C}(A'''A''C')$. Finally $B'' \in \mathcal{C}(B'A'''A'')$ and $\mathcal{C}(B'A'''A'') = \mathcal{C}(A'''A''C')$ so all six points lie on this circle, as required.

Theorem 6 *The centre of the second Lemoine circle is the midpoint of the line joining the Lemoine point to the centre of the ninepoint circle.*

Proof To be supplied by Sabin.

Theorem 7 *(SCHÖHILOG) The line from the midpoint of a side of a triangle to the midpoint of the altitude to the side that goes through the Lemoine point L (Figure 11).*

Proof(Rigby)

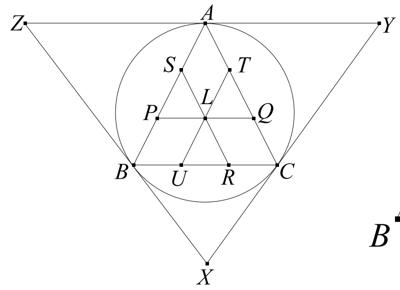

Figure 12:

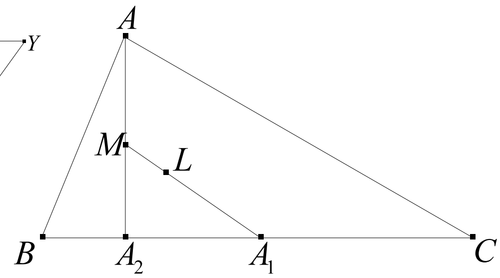

Figure 11:

Let the tangents to the circumcircle of the triangle ABC form a triangle XYZ as shown in Figure 12. Then AX is a symmedian and so L belongs to AX. Through L draw the lines PQ, RS and TU parallel to the sides YZ, ZX and XY, respectively.

First we claim that the six points P, U, R, Q, T, S lie on a circle with centre L (in fact, the first Lemoine circle of the triangle ABC).

$$\begin{aligned} \text{Since } \mathrm{P\widehat{Q}A} &= C\widehat{A}Y, \text{since PQ}\|\text{ZY} \\ &= A\widehat{C}Y, \text{since tangents}|\text{YC}| = |\text{YA}| \\ &= C\widehat{B}A, \text{angle between chord and tangent} \end{aligned}$$

then PQ is antiparallel to BC. Similarly show that SR antiparallel to AC and TU is antiparallel to AB. Claim now follows from theorem 4 above.

Since SR and TU are diameters of this circle, $STRU$ is a rectangle and the sides TR and SU are perpendicular to UR and so to BC. In particular the are parallel to the altitude through the vertex A.

Let AD be the altitude through A and let M be the midpoint of AD. Let F and G be the points where the lines MB and SU intersect and where MC and TR intersect (Figure 13).

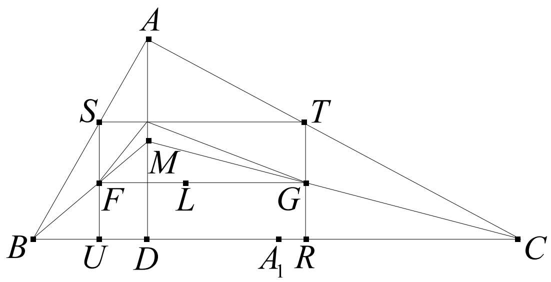

Figure 13:

Since $SU \| AD$ and M is the midpoint of AD and F is midpoint of SU. Similarly G is the midpoint of TR. Then the line FG passes through the centre of the circle containing the six points so L belongs to FG. Finally, FG is parallel to BC so the line joining M to A_1, the midpoint of BC, must intersect FG in its midpoint, i.e. point L.