Chapter 3. Inversion and Applications to Ptolemy and Euler

Power of a point with respect to a circle

Let A be a point and $\mathcal{C}$ a circle (Figure 1). If A is outside $\mathcal{C}$ and T is a point of contact of a tangent from A to $\mathcal{C}$, then for any secant from A with intersection points B_1, B_2 we have

$$|AT|^2 = |AB_1|.|AB_2|.$$

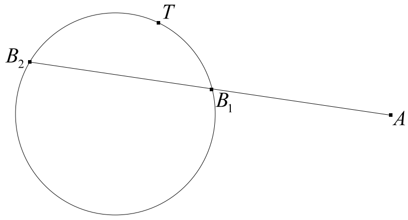

Figure 1:

This is defined to be the power of A with respect to the circle $\mathcal{C}$ and denoted by

$$\rho(A, \mathcal{C}) = |AT|^2$$

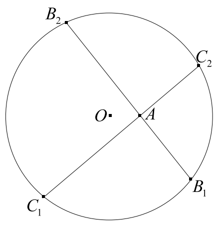

Figure 2:

Furthermore, if r is the radius of $\mathcal{C}$, and O its centre, then

$$\rho(A, \mathcal{C}) = |OA|^2 - r^2.$$

Now if A is interior to $\mathcal{C}$, then any two chords B_1B_2 and C_1C_2 intersecting at A (Figure 2) satisfy the property that

$$|B_1A||AB_2| = |C_1A||AC_2|$$

and if one of the chords goes through the centre O of $\mathcal{C}$, this common value can be shown to be

$$r^2 - |OA|^2$$

This is defined to be the power of A with respect to $\mathcal{C}$ if A is interior to $\mathcal{C}$.

Obviously, if A lies on the circle then $\rho(A, \mathcal{C}) = 0$

Inversion

Let A be a point in the plane ?P?. Then we define a mapping

$$Inv : \mathcal{P}\backslash\{A\} \rightarrow \mathcal{P}\backslash\{A\}$$

as follows. Let k be a positive, real number. Then a point B_2 is the image of a point B_1 under Inv (with respect to A and radius k) if B_2 lies along line joining A to B_1 and

$$|AB_1|.|AB_2| = k^2$$

We denote this mapping as

$$Inv(A, k^2)$$

Geometric Construction

To construct images of points P under inversion $Inv(A, k^2)$, one proceeds as follows.

First suppose that $|AP| < k$; thus P lies interior to the circle centred at A and having radius k. Let the chord through P and perpendicular to the line AP meet the circle at points T_1 and T_2. At T_1 and T_2 draw 2 tangents meeting at P' (Figure 3). Then

$$|AP||AP'| = k^2$$

This can be verified by observing that the triangle AT_1P and AT_1P' are similar.

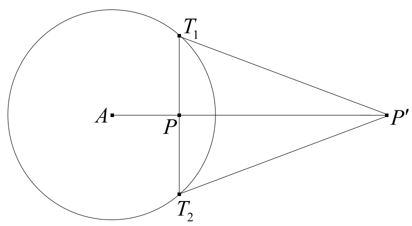

Figure 3:

Next suppose that $|AP| > k$; thus P lies outside the circle centred at A with radius k. Now draw a circle with AP as diameter and let it intersect the circle centred at A with radius k at the points T_1 and T_2 (Figure 4). Then P' is the point of intersection of the lines T_1T_2 and AP. To verify that $|AP'|.|AP| = k^2$, one again observes the triangles $AP'T_1$ and

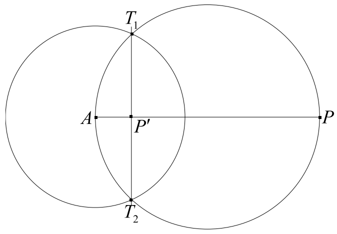

Figure 4:

APT_1 are similar.

We now state some properties of inversion mappings which we will use later in proving the theorems of Ptolemy and Euler.

Proposition 1 *Let A be a point, k a positive number and Inv the mapping $Inv(A, k^2)$. Then*

(a) if A belongs to a circle $\mathcal{C}(O, r)$ with centre O and radius r, then

$$Inv(\mathcal{C}(O,r)) \text{ is a line } l \text{ which is perpendicular to } OA.$$

(b) if l is a line which does not pass through A, then $Inv(l)$ is a circle such that l is perpendicular to the line joining A to the centre of the circle.

(c) if A does not belong to a circle $\mathcal{C}(O, r)$ then

$$Inv(\mathcal{C}(O,r)) = \mathcal{C}(O,r')$$
$$\text{with } r' = r\frac{k^2}{\rho(A,\mathcal{C}(O,r))}$$

(d) if $Inv(B_1) = B_2$ and $Inv(C_1) = C_2$ where B_1 and C_1 are two points in the plane, then

$$|B_2C_2| = |B_1C_1|.\frac{k^2}{|AB_1||AC_1|}$$

Remark Before discussing proofs of these properties of inversion, we explain the concept of antiparallel lines.

Begin with a triangle ABC. Then there are two ways to choose points D and E on the sides AB and AC so that the triangles ADE and ABC are similar.

In one case, the line DE is parallel to the side BC (Figure 5), and in

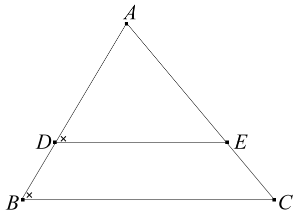

Figure 5:

the other case, $EDCB$ is a cyclic quadrilateral (Figure 6). We then say that

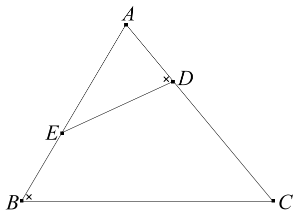

Figure 6:

the line segment DE is antiparallel to the side BC. In fact, a pair of opposite sides in any cyclic quadrilateral are said to be antiparallel to each other.

Proof of proposition

(a) Let $A \in \mathcal{C}(O, r)$, the circle with centre O and radius r. Let B_1 be a point on the other end of the diameter of $\mathcal{C}(O, r)$ containing A. Draw the line AB_1 through O and let B_2 be

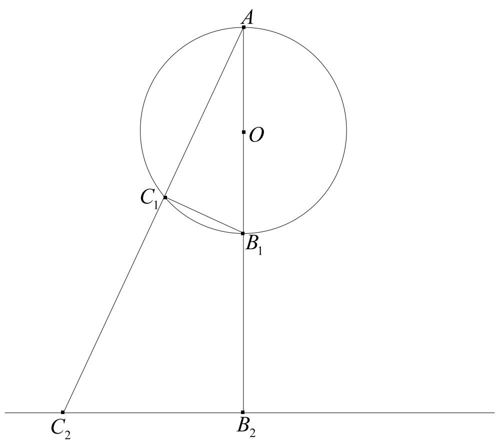

Figure 7:

image of B_1 under the inversion $Inv(A, k^2)$ (Figure 7).

Then

$$|AB_1||AB_2| = k^2.$$

Let C_1 be a point of $\mathcal{C}(O, r)$ distinct from B_1 and let $C_2 = Inv(C_1)$. Thus

$$|AC_1|.|AC_2| = k^2.$$

From this, the triangles AC_1B_1 and AC_2B_2 are similar and then

$$A\widehat{B}_2C_2 = A\widehat{C_1}B_1 = 90°.$$

Thus $Inv(\mathcal{C}(O, r))$ is the line through B_2 and perpendicular to AO.

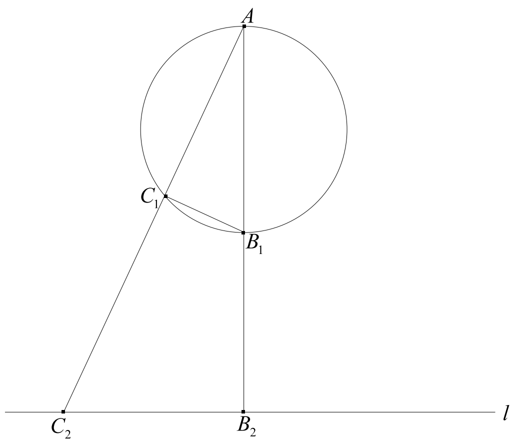

Figure 8:

(b) Let l be the line which does not include A (Figure 8). Drop a perpendicular from A to l meeting it at B_2. Let $B_1 = Inv(B_2)$ and we now claim that the circle with AB_1 as diameter is the image of l under Inv. Let C_2 be another point on l, and let $C_1 = Inv(C_2)$. Then since $|AC_1||AC_2| = |AB_1||AB_2|$, the triangles AC_1B_1 and AC_2B_2 are similar. Thus $A\widehat{C}_1B_1 = A\widehat{B}_2C_2 = 90°$, and so C_1 lies on the circle with AB_1 as diameter.

Remark: Note that if $Inv(B_1) = B_2$ and $Inv(C_1) = C_2$, then the line segments B_1C_1 and B_2C_2 are antiparallel.

We prove (d) before (c) as (c) is derived (in part) from (d).

(d) Let Inv be an inversion $Inv(A, k^2)$ for some k, and let (Figure 9)

$$Inv(B_1) = B_2$$
$$Inv(C_1) = C_2$$

Then B_1C_1 is antiparallel to B_2C_2 and the triangles AB_1C_1 and AC_2B_2 are similar. Thus

$$\frac{|B_2C_2|}{|B_1C_1|} = \frac{|AB_2|}{|AC_1|} = \frac{|AB_2|}{|AC_1|}.\frac{|AB_1|}{|AB_1|} = \frac{k^2}{|AB_1||AC_1|},$$
$$\text{or } |B_2C_2| = |B_1C_1|\frac{k^2}{|AB_1||AC_1|}$$

as required.

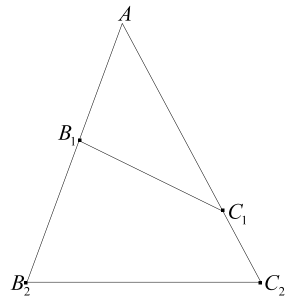

Figure 9:

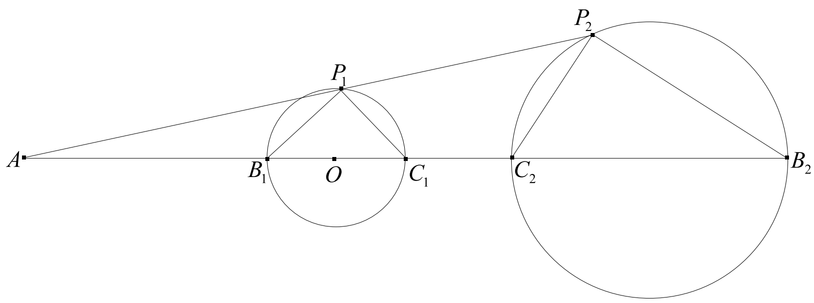

Figure 10:

(c) Let Inv be an inversion $Inv(A, k^2)$ and let O be the centre of a circle $\mathcal{C}(O, r)$ with radius r and not passing through A (Figure 10).

Let B_1 and C_1 be the points on the diameter of $\mathcal{C}(O, r)$ lying on the line AO and let

$$B_2 = Inv(B_1) \text{ and } C_2 = Inv(C_1).$$

Now choose a point P_1 on circle $\mathcal{C}(O, r)$. We claim that $P_2 = Inv(P_1)$ lies on the circle with C_2B_2 as diameter.

Since B_2P_2 is antiparallel to B_1P_1, then

$$B_2\widehat{P_2}P_1 = A\widehat{B_1}P_1,$$

and since P_2C_2 is antiparallel to P_1C_1, then

$$C_2\widehat{P_2}P_1 = A\widehat{C_1}P_1.$$

$$\begin{aligned} \text{Then } C_2\widehat{P_2}B_2 \quad &= P_1\widehat{P_2}B_2 - C_2\widehat{P_2}P_1 \\ &= A\widehat{B_1}P_1 - A\widehat{C_1}P_1 \\ &= (B_1\widehat{P_1}C_1 + B_1\widehat{C_1}P_1) - B_1\widehat{C_1}P_1 \\ &= B_1\widehat{P_1}C_1 = 90^\circ. \end{aligned}$$

Thus P_2 lies on the circle with C_2B_2 as diameter.

Finally, from (d) (just proved), we have

$$|B_2C_2| = |B_1C_1|\frac{k^2}{|AB_1||AC_1|} = \frac{k^2}{\rho(A, \mathcal{C}(O, r))}$$

But

$$|B_2C_2| = 2r' \text{ and } |B_1C_1| = 2r.$$

Thus

$$r' = r.\frac{k^2}{\rho(A, \mathcal{C}(O, r))}$$

as required.

<u>Applications</u>

We can use the above results to first prove Ptolemy's theorem.

Theorem 1 *Let ABCD be a cyclic quadrilateral (Figure 11). Then*

$$|AC||BD| = |AB||CD| + |AD||BC|$$

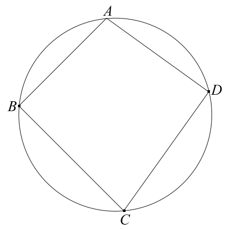

Figure 11:

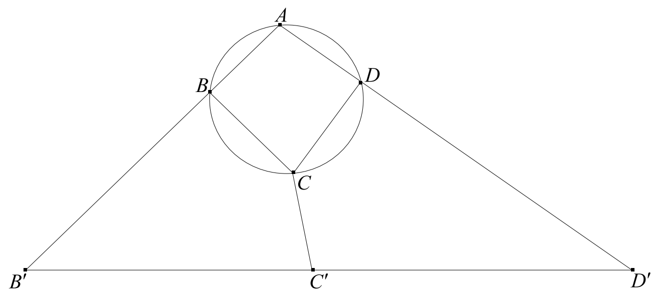

Figure 12:

Proof We take an inversion centred on A for some $k > 0$.

Let $B' = Inv(B), C' = Inv(C)$ and $D' = Inv(D)$ be the images (Figure 12). Since B, C and D lie on circle through A, the centre of the inversion mapping, then $B'C'D'$ are collinear, and furthermore from (d), we have

$$|B'C'| = |BC|\frac{k^2}{|AB|.|AC|}$$
$$|C'D'| = |CD|\frac{k^2}{|AC|.|AD|}$$
$$|B'D'| = |BD|\frac{k^2}{|AB|.|AD|}$$

But

$$|B'D'| = |B'C'| + |C'D'| \text{ so}$$
$$|BD|\frac{k^2}{|AB||AD|} = |BC|\frac{k^2}{|AB||AC|} + |CD|\frac{k^2}{|AC||AD|}$$

Multiplying across by $\dfrac{|AB||AC|.|AD|}{k^2}$, we get

$$|BD||AC| = |BC||AD| + |CD||AB|$$

as required. □

Another application is the theorem of Euler giving an expression for the distance between the circumcentre O and the incentre I of a triangle.

Theorem 2 ***(Euler)*** *Let ABC be a triangle with circumcentre O, circumradius R, incentre I and inradius r. Then*

$$|IO|^2 = R^2 - 2Rr$$

Proof Let I denote the incentre of the triangle ABC, let X, Y, A be the points of contact of the incircle with the sides BC, CA and AB respectively and, finally, let A', B', C' be the points of intersection of the lines joining I to the vertices and the sides of the triangle XYZ. The point A' lies on the line segment IA and similarly for B' and C' (Figure 13).

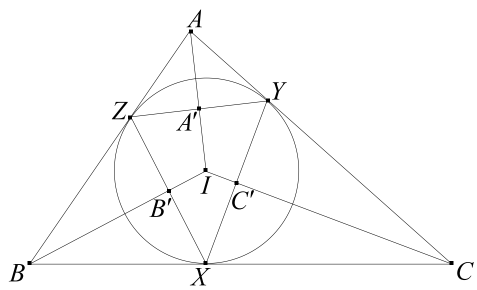

Figure 13:

The triangle $A'B'C'$ is the medial triangle of the triangle XYZ so the circumcircle of the triangle $A'B'C', \mathcal{C}(A'_1B'C')$ is the Euler circle of the triangle XYZ. Thus, if $R_{A'B'C'}$ denotes the radius of the circle $\mathcal{C}(A'_1B'C')$, then $R_{A'B'C'} = \frac{r}{2}$ where r is the radius of the incircle of the triangle ABC, i.e. the circumcircle of XYZ.

The triangles $IA'Y$ and IAY are similar so

$$\frac{|IA'|}{|IY|} = \frac{|IY|}{|IA|}$$
$$\text{i.e. } |IY|^2 = |IA'|.|IA|$$
$$\text{or } r^2 = |IA'||IA|$$

Similarly, we can show that

$$r^2 = |IB'||IB| = |IC'||IC|.$$

Now consider the Inversion mapping $Inv = Inv(I, r^2)$. Then $Inv(\mathcal{C}(A'B'C'))$ is a circle through ABC, i.e. $\mathcal{C}(ABC)$. Furthermore, from (c) above

$$\frac{r/2}{R} = \frac{r^2}{\rho(I, \mathcal{C}(ABC))}$$

Since I is internal to the circumcircle $\mathcal{C}(ABC)$ with radius R, then

$$\rho(I, \mathcal{C}(ABC)) = R^2 - |OI|^2.$$

Thus

$$R^2 - |OI|^2 = 2Rr$$

or

$$|OI|^2 = R^2 - 2Rr$$

as required. □

We get a similar result for enscribed circles.

Theorem 3 *Let ABC be a triangle and let $\mathcal{C}_a$ be the enscribed circle of this triangle with centre I_a, radius r_a. Then*

$$|OI_a|^2 = R^2 + 2Rr_a$$

where R is the radius of the circumcircle and O is its centre (Figure 14).

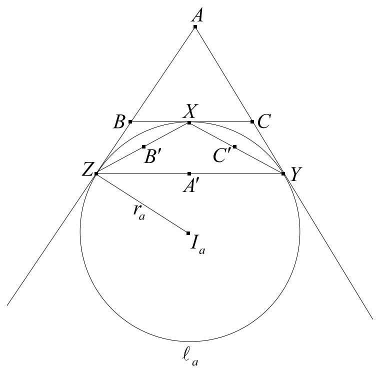

Figure 14:

Proof Let X, Y, Z be the points of contact of the circle $\mathcal{C}_a$ with sides BC, AC(extended) and AB(extended) respectively.

Let A', B', C' be points where I_aA and ZY intersect, I_aB and XZ intersect, and I_aC and XY intersect, respectively. Then $A'B'C'$ is the medial triangle of the triangle XYZ. Thus if $R_{A'B'C'}$ is the radius of the circumcircle of $A'B'C'$, then

$$R_{A'B'C'} = \frac{r_a}{2}.$$

We have $r_a^2 = |I_aA||I_aA'| = |I_aB||I_aB'| = |I_aC||I_aC'|$.
Now consider the inversion mapping $Inv = Inv(I_a, r_a^2)$, then

$$Inv(\mathcal{C}(ABC)) = \mathcal{C}(A'B'C')$$

$$\text{and so} \quad \frac{r_a/2}{R} = \frac{r_a^2}{\rho(I_a, \mathcal{C}(ABC))} = \frac{r_a^2}{|I_aO|^2 - R^2}$$

since I_a is exterior to $\mathcal{C}(ABC)$. Thus

$$2Rr_a = |I_aO|^2 - R^2 \text{ or } |OA_a|^2 = R^2 + 2Rr_a$$

as required.